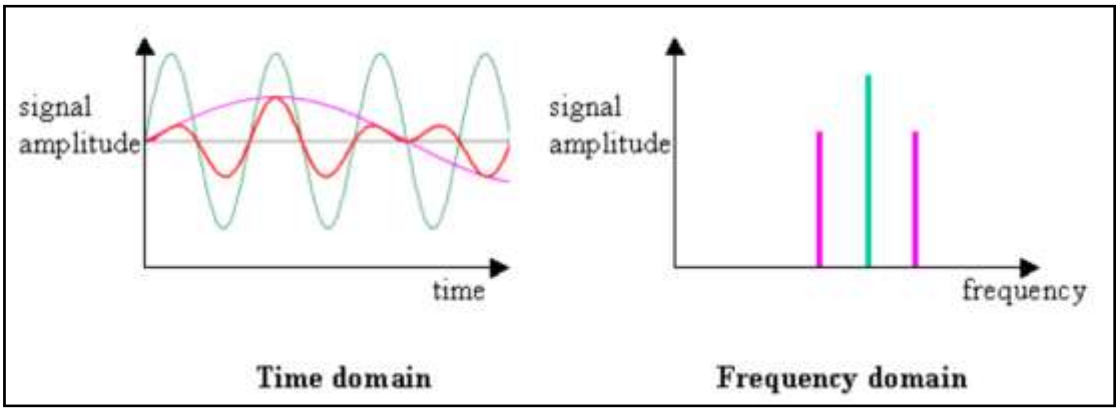
Biomedical Engineering
Engineering Analysis
Dr. Shahad Ali Hammood
shahad.ali@mustaqbal-college.edu.iq
10/11/2021

Lecture (3)

Fourier Transform: periodic, aperiodic signals and Special Function

3.1. The Fourier Transform is a mathematical technique that transforms a function of time, **f(t)**, to a function of frequency, **f(ω)**. It is closely related to the Fourier Series. The Fourier transform can be used to find the base frequencies that a wave is made of. The output of a Fourier transform is sometimes called a frequency spectrum or distribution because it displays a distribution of possible frequencies of the input.



3.2. Difference between Fourier Series and Fourier Transform

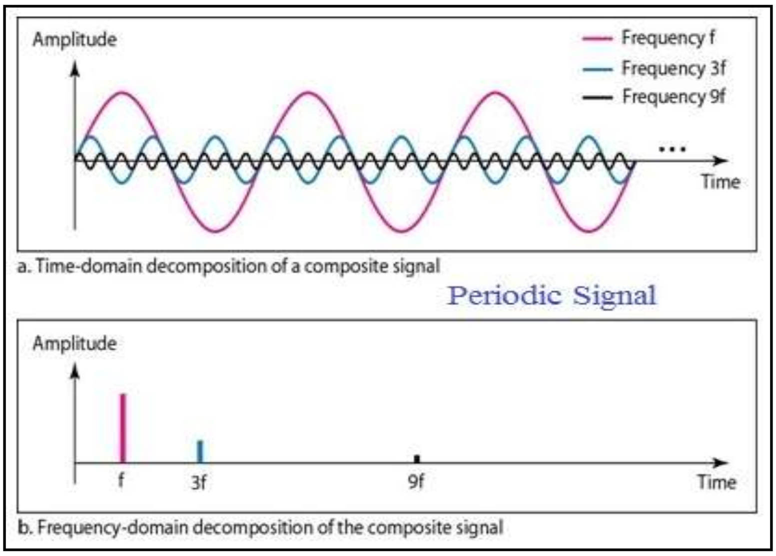
Fourier series is an expansion of periodic signal as a linear combination of sines and cosines while Fourier transform is the process or function used to convert signals from time domain in to frequency domain. Fourier series is defined for periodic signals and the **Fourier transform** can be applied to aperiodic (occurring without periodicity) signals. As mentioned above, the study of Fourier series actually provides motivation for the Fourier transform.

3.3. Periodic and Aperiodic Signals

3.3.1. Periodic Signal

Definition: A signal is considered to be **periodic signal** when it is repeated over cycle of time or regular interval of time. This means periodic signal repeats its pattern over a period. The function f(t) can be periodic if it satisfies following equation.

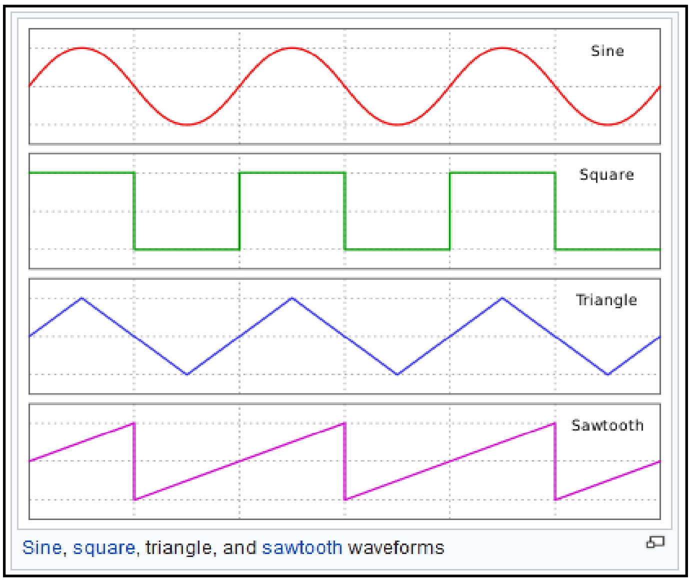
$f(t)= f(t+T)$



The frequency is rate of change with respect to time. The frequency and period are inverse of each other. Hence following can be implied.
f =1/T   (linear frequency) and T=1/f
ω=2πf   (angular frequency) and ω=2π/T
The units such as seconds(s), milliseconds(ms), microseconds(μs), and nanoseconds (ns) are used for time period while units such as Hz, KHz, MHz, and GHz are used for frequency.

3.3.1.1. Waveform of Periodic signal

There are four types (waveform) of periodic signal such as:



Sine, square, triangle, and sawtooth waveforms

For sine waveform T=2π-0= 2π sec.   , ω= 2π/2π=1 rad./sec.

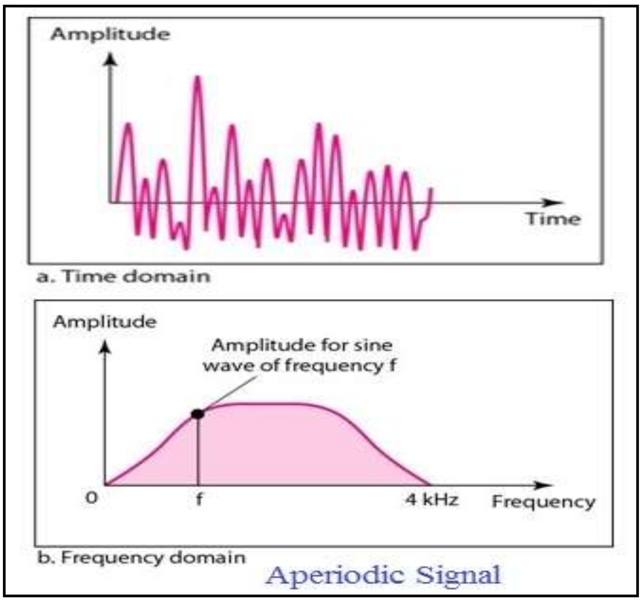
For triangular waveform T= 3-(-1)= 4 sec.   , ω= 2π/4= π/2 rad./sec.

For square waveform T= 2-0= 2 sec.   , ω= 2π/2= π rad./sec.

For sawtooth waveform T= 3-0= 3 sec.   , ω= 2π/3   rad./sec.

3.3.2. Aperiodic Signal (Non-periodic Signal)

Definition: A signal is considered to be **non-periodic or aperiodic** signal when it does not repeat its pattern over a period (i.e. interval of time).



Example 1. Determine whether the signal f(t)= cosωt is periodic or non- periodic?

Sol. f(t)= cosωt

We replace t ⟶ t+T

f(t+T)= cosω (t+T)= cos(ωt+ωT) , ω= 2π/T ⟶ ωT=2π

= cos(ωt+2π)

= cosωt. cos2π - sinωt. sin2π

= cosωt the result is equal to f(t) → the function is periodic

3.4. Fourier Transform Cases

1) The function is **not** even and **not** odd.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

2) The function is **even** (Fourier cosine transform).

$$F(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t).\cos(\omega t)\, dt$$

3) The function is **odd** (Fourier sine transform).

$$F(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t).\sin(\omega t)dt$$

Example 2. Find Fourier transform for f(t) = e^{-at} for t ≥ 0

Sol.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).e^{-i\omega t}.dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at}.e^{-i\omega t}.dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\omega)t}.dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_{0}^{\infty}$$

$$\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-\infty}}{-(a+i\omega)} - \frac{1}{-(a+i\omega)}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\omega}\right]$$

Example 3. Find the Fourier Cosine transform of

$$F(t)=\begin{cases} t & 0<t<1 \\ 2-t & 1<t<2 \\ 0 & t>2 \end{cases}$$

Sol.

$$F(\omega) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t).\cos(\omega t).dt$$

$$F(\omega) = \frac{2}{\sqrt{2\pi}}\left[\int_{0}^{1} t.\cos(\omega t).dt + \int_{1}^{2} (2-t).\cos(\omega t).dt + \int_{2}^{\infty} 0.\cos(\omega t).dt\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left[t.\frac{\sin\omega t}{\omega} + (1.\frac{\cos\omega t}{\omega^2})\right]_{0}^{1} + \left[\left((2-t).\left(\frac{\sin\omega t}{\omega}\right)\right) - (1.\frac{\cos\omega t}{\omega^2})\right]_{1}^{2}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left((1.\sin 1) + \left(1.\cos\frac{\omega}{\omega^2}\right)\right) - \left(\left(0 + \frac{\cos 0}{\omega^2}\right)\right)\right] + \left[\left((0) - \left(1.\frac{\cos 2\omega}{\omega^2}\right)\right) - \left((1.\sin 1) - \left(1.\cos\frac{\omega}{\omega^2}\right)\right)\right]$$

$$= \frac{2}{\sqrt{2\pi}}(\cos\frac{\omega}{\omega^2} - \frac{1}{\omega^2} - \cos\frac{2\omega}{\omega^2} + \cos\frac{\omega}{\omega^2}) = \frac{2}{\sqrt{2\pi}}(-\cos\frac{2\omega}{\omega^2} + 2\cos\frac{\omega}{\omega^2} - \frac{1}{\omega^2})$$

$$= -\frac{1}{\omega^2}\frac{2}{\sqrt{2\pi}}(\cos 2\omega - 2\cos\omega + 1)$$

3.5. Special functions

Special functions are **particular mathematical functions that have more or less established names and notations** due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications.

3.5.1. Types of Special Function

- **Gamma Function**
- **Unit Impulse function (The Dirac Delta Function)**
- **The Unit Step Function (Heaviside Function)**

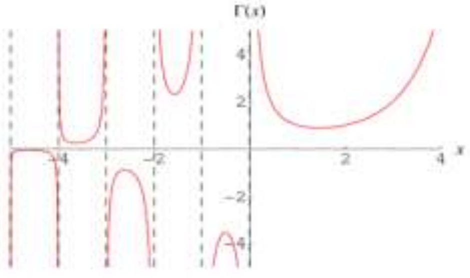


Figure 1: *Gamma Function*

3.5.1.1. Gamma Function

The Gamma function is defined by:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} . dx$$

Methods for solving the gamma function

if the coefficient is **positive number** $\longrightarrow$ $\Gamma(n) = (n-1)!$

If the coefficient is a **positive fraction** $\longrightarrow$ $\Gamma(n) = (n-1)\Gamma(n-1)$

If the coefficient is a **negative fraction** $\longrightarrow$ $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Examples:

$$\Gamma(5) = 4! = 4 \times 3 \times 2 \times 1 = 24$$

- ❖ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$$

Example. $\int_0^\infty \sqrt{x}.e^{-x^3} . dx$

Sol. 1) $-x^3 = -t \rightarrow x^3 = t \rightarrow x = t^{1/3}$

2) dx= $\frac{1}{3} t^{-2/3}. dt$

3) New limits: $x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

4) I= $\int_0^\infty t^{1/6}\, e^{-t} \left(\frac{1}{3}\, t^{-2/3}\, dt\right) = \frac{1}{3} \int_0^\infty t^{-1/2}\ e^{-t}.dt = \frac{1}{3}\, \Gamma \frac{1}{2} = \sqrt{\pi}/3$

Example. $\int_0^\infty x^4 . e^{-x^4}\ dx$

Sol. 1) let $-x^4 = -t \ \rightarrow\ x^4 = t \rightarrow x = t^{1/4}$

2) dx= $\frac{1}{4}\, t^{-3/4}.dt$

3) $at\ x = 0\ \rightarrow t = 0$

$at\ x = \infty \rightarrow t = \infty$

4) $I = \int_0^\infty t\, e^{-t} \left(\frac{1}{4}\, t^{-\frac{3}{4}}\, dt\right) = \frac{1}{4} \int_0^\infty t^{\frac{1}{4}}\ e^{-t}\ dt$

$= \frac{1}{4}\, \Gamma \frac{5}{4} = \frac{1}{16}\, \Gamma\, \frac{1}{4}$