Chapter seven

Application of integrals

7-1- Definite integrals:

If $f(x)$ is continuous in the interval $a \le x \le b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

1) $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$, *where c is constant.*

2) $\int_a^b \left(f(x) \mp g(x) \right)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) *Let* $a < c < b$ *then* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) *If* $f(x) \ge 0$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \ge 0$

7) *If* $f(x) \le g(x)$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

EX-1 **– Evaluate the following definite integrals:**

1) $\int_{2}^{6} \frac{dx}{x+2}$

2) $\int_{\pi/2}^{3\pi/2} \cos x \, dx$

3) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

4) $\int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

5) $\int_{-2}^{4} e^{-\frac{x}{2}} \, dx$

6) $\int_{0}^{\pi} (\pi - x) \cdot \cos x \, dx$

Sol. **–**

$$1)\ \int_{2}^{6} \frac{dx}{x+2} = \ln(x+2)\Big|_{2}^{6} = \ln(6+2) - \ln(2+2) = \ln 8 - \ln 4 = 3\ln 2 - 2\ln 2 = \ln 2$$

$$2)\ \int_{\pi/2}^{3\pi/2} \cos x \, dx = \sin x\Big|_{\pi/2}^{3\pi/2} = \sin(\frac{3}{2}\pi) - \sin(\frac{\pi}{2}) = -1 - 1 = -2$$

$$3)\ \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}\Big|_{-\sqrt{3}}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}(-\sqrt{3}) = \frac{\pi}{3} - (-\frac{\pi}{3}) = \frac{2}{3}\pi$$

$$4)\ \int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_{0}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$5)\ \int_{-2}^{4} e^{-\frac{x}{2}} \, dx = -2e^{-\frac{x}{2}}\Big|_{-2}^{4} = -2(e^{-2} - e) = 2(e - e^{-2})$$

$$6)\ Let \quad u = \pi - x \ \Rightarrow\ du = -dx \quad \& \quad dv = \cos x\,dx \ \Rightarrow\ v = \sin x$$

$$\int_{0}^{\pi} (\pi - x) \cdot \cos x \, dx = (\pi - x)\sin x\Big|_{0}^{\pi} + \int_{0}^{\pi} \sin x \, dx = (\pi - x)\sin x - \cos x\Big|_{0}^{\pi}$$

$$= (\pi - \pi)\sin\pi - \cos\pi - \big((\pi - 0)\sin 0 - \cos 0\big) = 0 - (-1) - (0 - 1) = 2$$

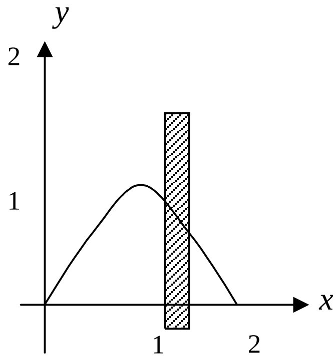
7-2- Area between two curves:

Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of x that are continuous for $a \le x \le b$ then the area bounded above by the y_1 curve, below by y_2 curve and on the sides by the vertical lines $x = a$ and $x = b$ is:-

$$A = \int_a^b \left[f_1(x) - f_2(x) \right] dx$$

***EX-2*- Find the area bounded by the *x*-axis and the curve:**

$$y = 2x - x^2$$



***Sol.*-**

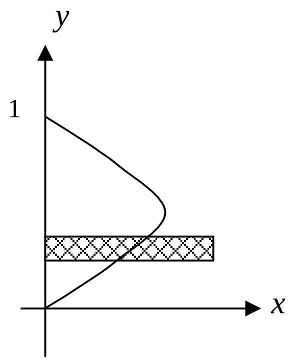
$$\left. \begin{array}{ll} y = 0 & \ldots\ldots\ldots\ldots(1) \\ y = 2x - x^2 & \ldots\ldots(2) \end{array} \right\} \Rightarrow \ x(x-2) = 0 \ \Rightarrow \ x = 0\ ,2$$

The points of the intersection of the curve and the *x*-axis are (0,0) and (2,0) then the area bounded by *x*-axis and the curve is:-

$$\int_0^2 (2x - x^2)\, dx = x^2 - \frac{x^3}{3} \bigg|_0^2 = 4 - \frac{8}{3} - (0-0) = \frac{4}{3}$$

***EX-3*- Find the area bounded by the *y*-axis and the curve:**

$$x = y^2 - y^3$$



***Sol.*-**

$$\left. \begin{array}{ll} x = 0 & \ldots\ldots\ldots\ldots(1) \\ x = y^2 - y^3 & \ldots\ldots(2) \end{array} \right\} \Rightarrow \ y^2(1-y) = 0 \ \Rightarrow \ y = 0\ ,1$$

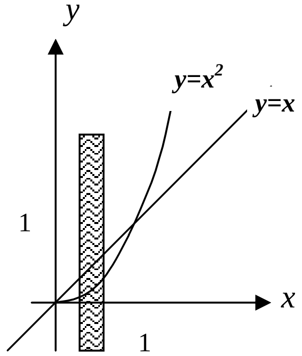
$$\Rightarrow \ intersection\ points\ \ (0,0),(0,1)$$

The area =

$$A = \int_0^1 (y^2 - y^3)\, dy = \frac{y^3}{3} - \frac{y^4}{4} \bigg|_0^1 = \frac{1}{3} - \frac{1}{4} - (0-0) = \frac{1}{12}$$

EX-4- **Find the area bounded by the curve** $y = x^2$ **and the line:** $y = x$

Sol.-



$$\left.\begin{array}{l} y = x^2 \text{ ......(1)} \\ y = x \text{ .....(2)} \end{array}\right\} \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$$

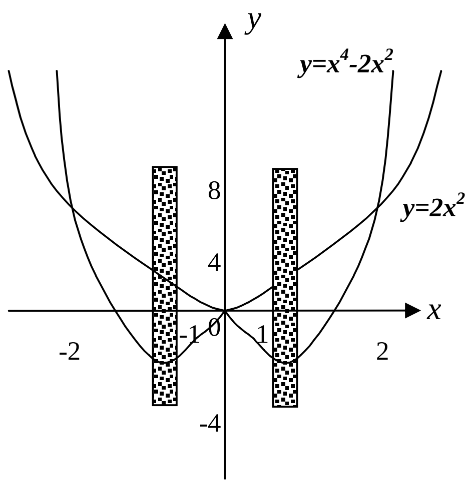
$$\Rightarrow \text{ intersection points } (0,0), (1,1)$$

The area =

$$A = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$$

EX-5- **Find the area bounded by the curves** $y = x^4 - 2x^2$ **and** $y = 2x^2$

Sol.-



$$\left.\begin{array}{l} y = x^4 - 2x^2 \text{ ......(1)} \\ y = 2x^2 \text{ ....(2)} \end{array}\right\} \Rightarrow x^2(x^2 - 4) = 0$$

$$\Rightarrow x = 0, 2, -2$$

$$\Rightarrow \text{ intersection points are } (0,0), (2,8), (-2,8)$$

The area =

$$A = \int_{-2}^{0} \left(2x^2 - (x^4 - 2x^2)\right)dx + \int_0^2 \left(2x^2 - (x^4 - 2x^2)\right)dx$$

$$= 2\int_0^2 \left(4x^2 - x^4\right)dx = 2\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 = 2\left[\frac{4}{3}\cdot 8 - \frac{32}{5} - 0\right]$$

$$= \frac{128}{15}$$

Notice:- We can use the double integration to calculate the area between two curves which bounded above by the curve $y = f_2(x)$ below by $y = f_1(x)$ on the left by the line $x = a$ and on the right by $x = b$, then:-

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx$$

To evaluate above integrals we follow:-

(a) integrating $\int dy$ with respect to y and evaluating the resulting integral the limits $y = f_1(x)$ and $y = f_2(x)$, then:

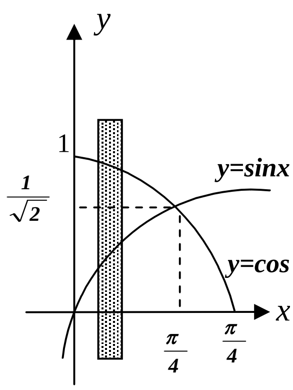
(b) integrating the result of (a) with respect to x between the limits $x = a$ and $x = b$.

If the area is bounded on the left by the curve $x = g_1(y)$, on the right by $x = g_2(y)$, below by the line $y = c$, and above by the line $y = d$, then it is better to integrate first with respect to x and then with respect to y. That is:-

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\, dy$$

EX-6- Find the area of the triangular region in the first quadrant bounded by the y-axis and the curve $y = \sin x$, $y = \cos x$.

Sol.-



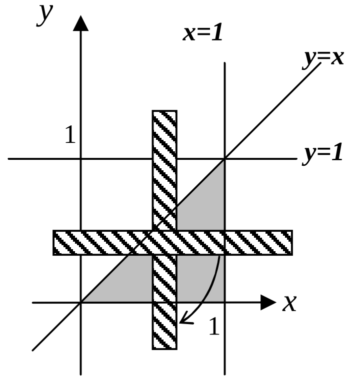
$$\left.\begin{aligned} y &= \sin x \quad .....(1) \\ y &= \cos x \quad .....(2) \end{aligned}\right\} \Rightarrow \sin x = \cos x \qquad \therefore x = \frac{\pi}{4}$$

The area =

$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\, dx = \int_0^{\frac{\pi}{4}} y \Big|_{\sin x}^{\cos x} dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) = \sqrt{2} - 1 = 0.414$$

EX-7- **Calculate:** $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$



Sol.- **We cannot solve the integration** $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$ **, hence we reverse the order of integration as follow:-**

$x = 1 \quad$ *and* $\quad y = 1$

$x = y \quad\quad\quad\quad y = 0$

$$A = \int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x}\,y\Big|_0^x\,dx = \int_0^1 \frac{\sin x}{x}(x-0)\,dx$$

$$= \int_0^1 \sin x\,dx = -\cos x\Big|_0^1 = -(\cos 1 - \cos 0) = 1 - \cos 1$$

EX-8- **Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.**

$1)\ \displaystyle\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\,dx \qquad\qquad 2)\ \int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-\frac{x}{2}}^{1-x} dy\,dx$

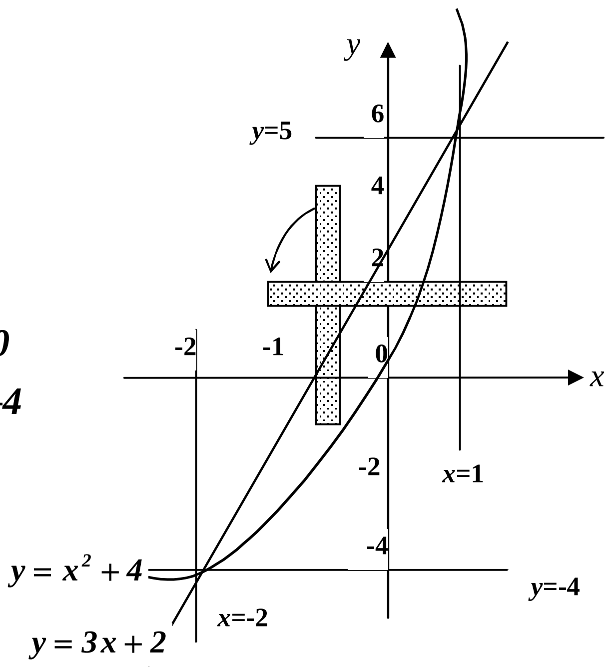
Sol.-

$$1)\quad \left.\begin{aligned} y &= 3x+2 \ \ .....(1) \\ y &= x^2+4x \ \ ...(2) \end{aligned}\right\} \Rightarrow$$

$$(x+2)(x-1)=0$$

$$\textit{either}\ \ x=-2 \Rightarrow y=-4$$

$$\textit{or}\ \ \ x=1 \Rightarrow y=5$$



$$(a)\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\,dx=\int_{-2}^{1} y\Big|_{x^2+4x}^{3x+2} dx=\int_{-2}^{1}(2-x-x^2)dx$$

$$=2x-\frac{x^2}{2}-\frac{x^3}{3}\Big|_{-2}^{1}=2-\frac{1}{2}-\frac{1}{3}-(-4-2+\frac{8}{3})=\frac{9}{2}$$

(b) *The reversed integral is :-*

$$y=3x+2 \Rightarrow x=\frac{y-2}{3}$$

$$y=x^2+4x \Rightarrow (x+2)^2=y+4 \Rightarrow x=-2\mp\sqrt{y+4}$$

$$\text{Since } -2\le x\le 1 \Rightarrow x=-2+\sqrt{y+4}$$

$$\int_{-4}^{5}\int_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dx\,dy=\int_{-4}^{5} x\Big|_{\frac{y-2}{3}}^{-2+\sqrt{y+4}}=\int_{-4}^{5}\left(-2+\sqrt{y+4}-\frac{y-2}{3}\right)dy$$

$$=-2y+\frac{2}{3}(y+4)^{3/2}-\frac{(y-2)^2}{6}\Big|_{-4}^{5}$$

$$=-10+\frac{2}{3}(27)-\frac{9}{6}-(8+0-\frac{36}{6})=\frac{9}{2}$$

$=$ *The same result as in (a).*

$$2)\ (a)\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx+\int_{0}^{2}\int_{-\frac{x}{2}}^{1-x} dy\,dx=\int_{-1}^{0} y\Big|_{-2x}^{1-x} dx+\int_{0}^{2} y\Big|_{-\frac{x}{2}}^{1-x} dx$$

$$=\int_{-1}^{0}(1+x)\,dx+\int_{0}^{2}(1-\frac{x}{2})\,dx=x+\frac{x^2}{2}\Big|_{-1}^{0}+x-\frac{x^2}{4}\Big|_{0}^{2}$$

$$=0-(-1+\frac{1}{2})+2-1-0=\frac{3}{2}$$

(b) 1st region

$$\left.\begin{array}{l} y=1-x\ .....(1)\\ y=-2x\ ......(2)\end{array}\right\}\Rightarrow x=-1\Rightarrow y=2 \qquad x \text{ from } -1 \text{ to } 0$$

2nd region

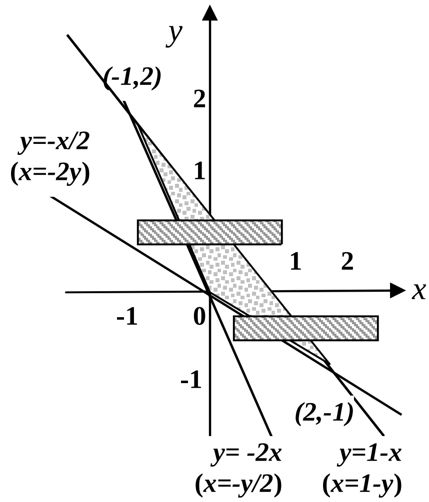
$$\left.\begin{aligned} y &= 1 - x \quad ...(1) \\ y &= -\frac{x}{2} \quad .....(2) \end{aligned}\right\} \Rightarrow x = 2 \Rightarrow y = -1 \qquad y \;\; from \;\; 0 \;\; to \;\; 2$$

$$\int_{0}^{2}\int_{-\frac{y}{2}}^{1-y} dx\,dy + \int_{-1}^{0}\int_{-2y}^{1-y} dx\,dy = \int_{0}^{2} x \Big|_{-\frac{y}{2}}^{1-y} dy + \int_{-1}^{0} x \Big|_{-2y}^{1-y} dy$$

$$= \int_{0}^{2} (1 - \frac{y}{2})dy + \int_{-1}^{0} (1 + y)dy = y - \frac{y^2}{4}\Big|_{0}^{2} + y + \frac{y^2}{2}\Big|_{-1}^{0}$$

$$= 2 - 1 - 0 + 0 - (-1 + \frac{1}{2}) = \frac{3}{2}$$

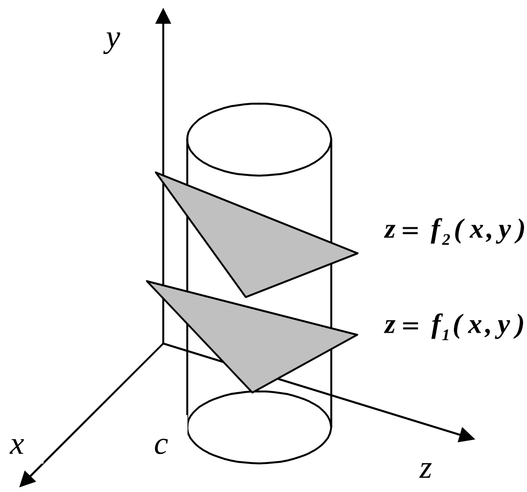
$= The\ same\ result\ as\ in\ \ (a).$



7-3- <u>Triple integrals (Volume)</u>:

Consider a region N in xyz-space bounded below by a surface $z = f_1(x,y)$, above by the surface $z = f_2(x,y)$ and laterally by a cylinder c with elements parallel to the z-axis. Let A denote the region of the xy-plane enclosed by cylinder c (that is, A is the region covered by the orthogonal projection of the solid into xy-plane). Then the volume V of the region V can be found by evaluating the triply iterated integral:-

$$V = \iint_{A} \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$$



Let z-limits of integration indicate that for every (x,y) in the region A,Z may extend from the lower surface $z = f_1(x,y)$ to the surface $z = f_2(x,y)$. The y- and x-limits of integration have not been given explicitly in equation above, but are indicated as extending over the region A.

We can find the equation of the boundary of the region A by eliminating z between the two equations $z = f_1(x,y)$ and $z = f_2(x,y)$, thus obtaining an equation $f_1(x,y) = f_2(x,y)$ which contains no z, and interpret it as an equation in the xy-plane.

EX-9 **The volume in the first octant bounded by the cylinder $x = 4 - y^2$, and the planes $z = y,\ x = 0,\ z = 0.$**

Sol.-

$$x = 4 - y^2 \ \Rightarrow\ y = \mp\sqrt{4 - x} \qquad in\ first\ octant : -$$

$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^y dz\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} z\Big|_0^y dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} y\, dy\, dx = \int_0^4 \frac{y^2}{2}\Bigg|_0^{\sqrt{4-x}} dx$$

$$= \frac{1}{2}\int_0^4 (4 - x - 0)dx = \frac{1}{2}\left[4x - \frac{x^2}{2}\right]_0^4 = \frac{1}{2}\left[16 - \frac{16}{2} - 0\right] = 4$$

EX-10 **The volume enclosed by the cylinders $z = 5 - x^2, z = 4x^2$ and the planes $y = 0,\ \ x + y = 1$.**

Sol.-

$$\left.\begin{aligned} z &= 5 - x^2 \ ...(1) \\ z &= 4x^2 \ ......(2) \end{aligned}\right\} \Rightarrow\ x = \mp 1$$

$$V = \int_{-1}^{1} \int_0^{1-x} \int_{4x^2}^{5-x^2} dz\, dy\, dx = \int_{-1}^{1} \int_0^{1-x} z\Big|_{4x^2}^{5-x^2} dy\, dx = \int_{-1}^{1} \int_0^{1-x} (5 - 5x^2)\, dy\, dx$$

$$= 5\int_{-1}^{1} (1 - x^2)\, y\Big|_0^{1-x} dx = 5\int_{-1}^{1} (1 - x^2)(1 - x)dx$$

$$= 5\int_{-1}^{1} (1 - x - x^2 + x^3)\, dx = 5\left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= 5\left[(1+1) - \frac{1}{2}(1-1) - \frac{1}{3}(1+1) + \frac{1}{4}(1-1)\right] = \frac{20}{3}$$

EX-11 The volume enclosed by the cylinders $y^2 + 4z^2 = 16$ and the planes $x = 0$, $x + y = 4$.

Sol.-

$$y^2 + 4z^2 = 16 \quad \Rightarrow \quad y = \mp 2\sqrt{4 - z^2}$$

$$V = \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} \int_{0}^{4-y} dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} (4 - y)\, dy\, dz = \int_{-2}^{2} 4y - \frac{y^2}{2} \Bigg|_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} dz = 16 \int_{-2}^{2} \left(4 - z^2\right)^{1/2} dz$$

$$let \quad z = 2\sin\theta \;\Rightarrow\; dz = 2\cos\theta\, d\theta \;,\quad \theta = \sin^{-1}\frac{z}{2} \quad \underset{at\ z=2\ \Rightarrow\ \theta=\frac{\pi}{2}}{\overset{at\ z=2\ \Rightarrow\ \theta=\frac{\pi}{2}}{\Rightarrow\Rightarrow\Rightarrow}}$$

$$V = 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 - 4\sin^2\theta)^{1/2}\, 2\cos\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 32\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 32\left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{1}{2}(0 - 0)\right] = 32\pi$$

EX-12 The volume bounded by the ellipse paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

Sol.-

$$\left.\begin{array}{l} z = 18 - x^2 - 9y^2 \;\; ..(1) \\ z = x^2 + 9y^2 \;\; ....(2) \end{array}\right\} \Rightarrow \; 9 - x^2 - 9y^2 = 0 \;\Rightarrow\; y = \mp\frac{1}{3}\sqrt{9 - x^2}$$

$$V = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz\, dy\, dx = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \left[18 - x^2 - 9y^2 - (x^2 + 9y^2)\right] dy\, dx$$

$$V = 2\int_{-3}^{3} (9-x^2)y - 3y^3 \Big]_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} dx$$

$$= 2\int_{-3}^{3}\left[(9-x^2)\left(\frac{\sqrt{9-x^2}}{3}+\frac{\sqrt{9-x^2}}{3}\right)-3\left(\frac{(9-x^2)^{3/2}}{27}+\frac{(9-x^2)^{3/2}}{27}\right)\right]dx$$

$$= \frac{8}{9}\int_{-3}^{3}(9-x^2)^{3/2}dx$$

$$let\ \ x = 3sin\theta \Rightarrow dx = 3cos\theta\, d\theta \quad , \quad \theta = sin^{-1}\frac{x}{3} \ \overset{at\ x=3\ \Rightarrow\ \theta=\frac{\pi}{2}}{\underset{at\ x=-3\ \Rightarrow\ \theta=-\frac{\pi}{2}}{\Rightarrow\Rightarrow\Rightarrow}}$$

$$= \frac{8}{9}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(9-9\sin^2\theta)^{3/2}\,3cos\theta\, d\theta = 72\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^4\theta\, d\theta = 72\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{1+cos2\theta}{2})^2\, d\theta$$

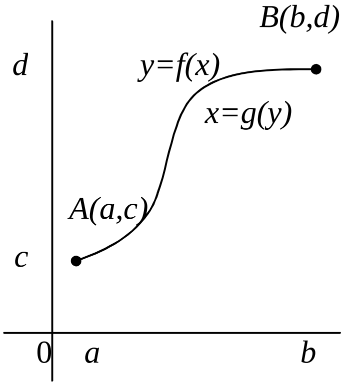
$$= 18\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2cos2\theta+cos^2 2\theta)\, d\theta = 18\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2cos2\theta+\frac{cos4\theta}{2})\, d\theta$$

$$= 9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+4cos2\theta+cos4\theta)\, d\theta = 9\left[3\theta+2\sin 2\theta+\frac{1}{4}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 9\left[3(\frac{\pi}{2}+\frac{\pi}{2})+2(sin\pi-sin(-\pi))+\frac{1}{4}(sin2\pi-sin(-2\pi))\right] = 27\pi$$

7-4- <u>The length of a plane curve:-</u>

The length of the curve $y = f(x)$ **from point** $A(a,c)$ **to** $B(b,d)$ **is:-**



$$L = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}\, dx$$

If x **can be expressed as a function of** y **then the length is:-**

$$L = \int_c^d \sqrt{1+(\frac{dx}{dy})^2}\, dy$$

Let the equation of motion be $x = g(t)$ and $y = h(t)$ continuously differentiable for t between $t_a(at\ A)\ and\ t_b(at\ B)$, then the length of the curve is:-

$$L = \int_{t_a}^{t_b} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

EX-13 **– Find the length of the curve:**

$1)\ y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$

$2)\ 9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3},3)$

$3)\ y = x^{\frac{2}{3}}$ from $x = -1$ to $x = 8$

Sol.

$$1)\quad y = \frac{1}{3}(x^2+2)^{\frac{3}{2}} \quad \Rightarrow \quad \frac{dy}{dx} = x(x^2+2)^{\frac{1}{2}}$$

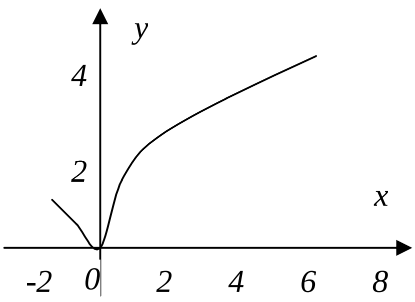
$$L = \int_0^3 \sqrt{1 + x^2(x^2+2)}\ dx = \int_0^3 (x^2+1)\ dx = \frac{x^3}{3} + x\Big|_0^3 = 9 + 3 - 0 = 12$$

$$2)\ 9x^2 = 4y^3 \quad \Rightarrow \quad x = \mp\frac{2}{3}y^{\frac{3}{2}} \quad Since\ \ x\ \ from\ \ 0\ \ to\ \ 2\sqrt{3}$$

$$then \quad x = \frac{2}{3}y^{\frac{3}{2}} \quad \Rightarrow \quad \frac{dx}{dy} = y^{\frac{1}{2}}$$

$$L = \int_0^3 \sqrt{1+y}\ dy = \frac{2}{3}(1+y)^{\frac{3}{2}}\Big|_0^3 = \frac{2}{3}[8-1] = \frac{14}{3}$$

$$3)\ y = x^{\frac{2}{3}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$$



$$Since \quad \frac{dy}{dx} = \infty \quad at \quad x = 0$$

$$then \quad x = \mp y^{\frac{3}{2}} \Rightarrow \frac{dx}{dy} = \mp\frac{3}{2}y^{\frac{1}{2}}$$

$$L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\ dy + \int_0^4 \sqrt{1+\frac{9}{4}y}\ dy = \frac{1}{18}\left[\frac{(4+9y)^{\frac{3}{2}}}{3/2}\Big|_0^1 + \frac{(4+9y)^{\frac{3}{2}}}{3/2}\Big|_0^4\right]$$

$$= \frac{1}{27}\left[(13\sqrt{13} - 4\sqrt{4}) + (40\sqrt{40} - 4\sqrt{4})\right] = 10.51$$

EX-14 **– Find the distance traveled between** $t=0$ **and** $t=\frac{\pi}{2}$ **a particle** ***P(x,y)*** **whose position at time** t **is given by:-** $x=a\cos t+a\cdot t\sin t \quad and \quad y=a\sin t-a\cdot t\cos t$ **where** a **is a positive constant.**

Sol.

$$x=a\cos t+a\cdot t\sin t \;\Rightarrow\; \frac{dx}{dt}=a\cdot t\cos t$$

$$y=a\sin t-a\cdot t\cos t \;\Rightarrow\; \frac{dy}{dt}=a\cdot t\sin t$$

$$L=\int_a^b\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\int_0^{\frac{\pi}{2}}\sqrt{a^2\cdot t^2\cos^2 t+a^2\cdot t^2\sin^2 t}\,dt$$

$$=a\int_0^{\frac{\pi}{2}}t\,dt=\frac{a}{2}t^2\Big|_0^{\frac{\pi}{2}}=\frac{a}{2}\left[\frac{\pi^2}{4}-0\right]=\frac{a}{8}\pi^2$$

EX-15 **– Find the length of the curve:-**

$$x=t-\sin t \quad and \quad y=1-\cos t \;;\; 0\le t\le 2\pi$$

Sol.

$$x=t-\sin t \;\Rightarrow\; \frac{dx}{dt}=1-\cos t$$

$$y=1-\cos t \;\Rightarrow\; \frac{dy}{dt}=\sin t$$

$$L=\int_a^b\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\int_0^{2\pi}\sqrt{(1-\cos t)^2+\sin^2 t}\,dt$$

$$=\int_0^{2\pi}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt=\int_0^{2\pi}\sqrt{1-2\cos t+1}\,dt$$

$$=2\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt=2\int_0^{2\pi}\sin\frac{t}{2}\,dt=-4\cos\frac{t}{2}\Big|_0^{2\pi}$$

$$=-4[\cos\pi-\cos 0]=-4[-1-1]=8$$

7-5- The surface area:

Suppose that the curve $y = f(x)$ is rotated about the x-axis. It will generate a surface in space. Then the surface area of the shape is:-

$$S = \int_a^b 2\pi \, y \sqrt{1 + (\frac{dy}{dx})^2} \, dx$$

If the curve rotated about the y-axis, then the surface area is:-

$$S = \int_c^d 2\pi \, x \sqrt{1 + (\frac{dx}{dy})^2} \, dy$$

If the curve sweeps out the surface is given in parametric form with x and y as functions of a third variable t that varies from t_a to t_b then we may compute the surface area from the formula:-

$$S = \int_{t_a}^{t_b} 2\pi \, \rho \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt$$

where ρ is the distance from the axis of revolution to the element of arc length and is expressed as a function of t.

EX-16 – The circle $x^2 + y^2 = r^2$ is revolved about the x-axis. Find the area of the sphere generated.

Sol.-

$$y = \sqrt{r^2 - x^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$S = \int_a^b 2\pi \, y \sqrt{1 + (\frac{dy}{dx})^2} \, dx = \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = 2\pi \, r \int_{-r}^{r} dx$$

$$= 2\pi \, r \, x \Big|_{-r}^{r} = 2\pi \, r\big(r - (-r)\big) = 4\pi \, r^2$$

EX-17 – Find the area of the surface generated by rotating the portion of the curve $y=\frac{1}{3}(x^2+2)^{\frac{3}{2}}$ between $x=0$ and $x=3$ about the y-axis.

Sol.-

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \Rightarrow x=((3y)^{\frac{2}{3}}-2)^{\frac{1}{2}} \Rightarrow \frac{dx}{dy}=\frac{1}{(3y)^{\frac{1}{3}}\cdot((3y)^{\frac{2}{3}}-2)^{\frac{1}{2}}}$$

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \quad \overset{at\ x=0}{\Rightarrow\Rightarrow\Rightarrow} \quad y=\frac{2\sqrt{2}}{3} \quad and \quad \overset{at\ x=3}{\Rightarrow\Rightarrow\Rightarrow} \quad y=\frac{11\sqrt{11}}{3}$$

$$S=\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} 2\pi\sqrt{(3y)^{\frac{2}{3}}-2}\cdot\sqrt{1+\frac{1}{((3y)^{\frac{2}{3}}-2)(3y)^{\frac{2}{3}}}}\,dy$$

$$=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\sqrt{\frac{(3y)^{\frac{4}{3}}-2(3y)^{\frac{2}{3}}+1}{(3y)^{\frac{2}{3}}}}\,dy=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\sqrt{\frac{((3y)^{\frac{2}{3}}-1)^2}{(3y)^{\frac{2}{3}}}}\,dy$$

$$=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\left[(3y)^{\frac{1}{3}}-(3y)^{-\frac{1}{3}}\right]dy=2\pi\left[\frac{1}{3}\frac{(3y)^{\frac{4}{3}}}{\frac{4}{3}}-\frac{1}{3}\frac{(3y)^{\frac{2}{3}}}{\frac{2}{3}}\right]_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}$$

$$=\pi\left[\frac{(3\cdot\frac{11\sqrt{11}}{3})^{\frac{4}{3}}}{2}-(3\cdot\frac{11\sqrt{11}}{3})^{\frac{2}{3}}-\frac{(3\cdot\frac{2\sqrt{2}}{3})^{\frac{4}{3}}}{2}-(3\cdot\frac{2\sqrt{2}}{3})^{\frac{2}{3}}\right]=\frac{99}{2}\pi$$

EX-18 – The arc of the curve $y=\frac{x^3}{3}+\frac{1}{4x}$ from $x=1$ to $x=3$ is rotated about the line $y=-1$. Find the surface area generated.

Sol.-

$$y=\frac{x^3}{3}+\frac{1}{4x} \quad \Rightarrow \quad \frac{dy}{dx}=x^2-\frac{1}{4x^2}=\frac{4x^4-1}{4x^2}$$

$$S=2\pi\int_1^3\left(\frac{x^3}{3}+\frac{1}{4x}+1\right)\sqrt{1+\frac{(4x^4-1)^2}{16x^4}}\,dx$$

$$=2\pi\int_1^3\frac{4x^4+12x+3}{12x}\sqrt{\frac{(4x^4+1)^2}{16x^4}}\,dx$$

$$=\frac{\pi}{24}\int_1^3(16x^5+48x^2+16x+12x^{-2}+3x^{-3})dx$$

$$=\frac{\pi}{24}\left[\frac{8}{3}x^6+16x^3+8x^2-\frac{12}{x}-\frac{3}{2x^2}\right]_1^3$$

$$=\frac{\pi}{24}\left[\frac{8}{3}(729-1)+16(27-1)+8(9-1)-12(\frac{1}{3}-1)-\frac{3}{2}(\frac{1}{9}-1)\right]$$

$$=\frac{1823}{18}\pi$$

_EX-19_ – Find the area of the surface generated by rotating the curve $x=t^2 \quad , \quad y=t \quad , \quad 0\le t\le 1$ **about the** x**-axis.**

Sol.-

$$x=t^2 \quad \Rightarrow \quad \frac{dx}{dt}=2t \quad \text{and} \quad y=t \quad \Rightarrow \quad \frac{dy}{dt}=1$$

$$S=\int_{t_a}^{t_b}2\pi\,\rho\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=2\pi\int_0^1 t\sqrt{4t^2+1}\,dt$$

$$=\frac{\pi}{4}\left[\frac{(4t^2+1)^{\frac{3}{2}}}{3/2}\right]_0^1=\frac{\pi}{6}\left[5\sqrt{5}-1\right]$$

Problems – 7

1) **Find the area of the region bounded by the given curves and lines for the following problems:-**

1. **The coordinate axes and the line** $x + y = a$
2. **The** x**-axis and the curve** $y = e^x$ **and the lines** $x = 0$, $x = 1$
3. **The curve** $y^2 + x = 0$ **and the line** $y = x + 2$
4. **The curves** $x = y^2$ **and** $x = 2y - y^2$
5. **The parabola** $x = y - y^2$ **and the line** $x + y = 0$

$$(ans.:\ 1.\frac{a^2}{2}\ ;\ 2.e - 1\ ;\ 3.\frac{9}{2}\ ;\ 4.\frac{1}{3}\ ;\ 5.\frac{4}{3})$$

2) **Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.**

1. $\int_0^2 \int_1^{e^x} dy\,dx$ $\qquad$ $(ans.:\ \int_1^{e^2} \int_{\ln y}^{2} dx\,dy\ ;\ e^2 - 3)$

2. $\int_0^1 \int_{\sqrt{y}}^{1} dx\,dy$ $\qquad$ $(ans.:\ \int_0^1 \int_0^{x^2} dy\,dx\ ;\ \frac{1}{3})$

3. $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$ $\qquad$ $(ans.:\ \int_{-2}^{2} \int_0^{\sqrt{\frac{4-x^2}{2}}} y\,dy\,dx\ ;\ \frac{8}{3})$

3) **Find the volume of the tetrahedron bounded by the plane** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **and the coordinate planes.**

$$(ans.:\ \frac{1}{6}|abc|)$$

4) **Find the volume bounded by the plane** $z = 0$ **laterally by the elliptic cylinder** $x^2 + 4y^2 = 4$ **and above by the plane** $z = x + 2$ **.**

$$(ans.:\ 4\pi)$$

5) Find the lengths of the following curves:-

1. $y = x^{\frac{3}{2}}$ *from* $(0,0)$ *to* $(4,8)$ (ans.: $\frac{8}{27}(10\sqrt{10}-1)$)

2. $y = \frac{x^3}{3} + \frac{1}{4x}$ *from* $x = 1$ *to* $x = 3$ (ans.: $\frac{53}{6}$)

3. $x = \frac{y^4}{4} + \frac{1}{8y^2}$ *from* $y = 1$ *to* $y = 2$ (ans.: $\frac{123}{32}$)

4. $(y+1)^2 = 4x^3$ *from* $x = 0$ *to* $x = 1$ (ans.: $\frac{4}{27}(10\sqrt{10}-1)$)

6) Find the distance traveled by the particle *P(x,y)* between *t=0* and *t=4* if the position at time *t* is given by: $x = \frac{t^2}{2}$; $y = \frac{1}{3}(2t+1)^{\frac{3}{2}}$

(ans.: 12)

7) The position of a particle *P(x,y)* at time *t* is given by: $x = \frac{1}{3}(2t+3)^{\frac{3}{2}}$; $y = \frac{t^2}{2} + t$ **. Find the distance it travel between *t=0* and *t=3*.**

(ans.: $\frac{21}{2}$)

8) Find the area of the surface generated by rotating about the *x*-axis the arc of the curve $y = x^3$ *between* $x = 0$ *and* $x = 1$.

(ans.: $\frac{\pi}{27}(10\sqrt{10}-1)$)

9) Find the area of the surface generated by rotating about the *y*-axis the arc of the curve $y = x^2$ *between* $(0,0)$ *and* $(2,4)$.

(ans.: $\frac{\pi}{6}(17\sqrt{17}-1)$)

10) Find the area of the surface generated by rotating about the *y*-axis the curve $y = \frac{x^2}{2} + \frac{1}{2}$; $0 \le x \le 1$. (ans.: $\frac{2}{3}\pi(2\sqrt{2}-1)$)

11) The curve described by the particle *P(x,y)* $x = t + 1$, $y = \frac{t^2}{2} + t$ **from** $t = 0$ **to** $t = 4$ **is rotated about the *y*-axis. Find the surface area that is generated.**

(ans.: $\frac{2\sqrt{2}}{3}\pi(13\sqrt{13}-1)$)