Al-Mustaqbal University College

Department of Medical Instrumentation Techniques Engineering

Mathematic I

LECTURE FIVE

Integration

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هندسة تقنيات الاجهزه الطبيه/ المرحله الأولى

Hyperbolic functions:

The hyperbolic functions are a special combination of the functions e^x and e^{-x}.

Definitions:

1- $\sinh x = \frac{e^x - e^{-x}}{2}$

2- $\cosh x = \frac{e^x + e^{-x}}{2}$

3- $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4- $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}$

5- $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

6- $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Some important relations and identities:

1) $\cosh^2 x - \sinh^2 x = 1$
2) $\tanh^2 x + \coth^2 x = 1$
3) $\coth^2 x - \operatorname{csch}^2 x = 1$
4) $\sinh(-x) = -\sinh x$
5) $\coth(-x) = \coth x$
6) $\tanh(-x) = -\tanh x$
7) $\sinh x \pm \cosh x = \pm e^{\pm x}$
8) $\sinh(x \pm y) = \sinh y \cosh x \pm sinh\ x \cosh y$
9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
10) $\sinh 2x = 2 \sinh x cosh\ x$

11) $sinh^2 x = \frac{\cosh(2x)-1}{2}$

12) $cosh^2 x = \frac{\cosh(2x)+1}{2}$

Derivative of hyperbolic functions:

1) $y = \sinh x \rightarrow \frac{dy}{dx} = \cosh x$

2) $y = \cosh x \rightarrow \frac{dy}{dx} = \sinh x$

3) $y = \tanh x \rightarrow \frac{dy}{dx} = \text{sech}^2 x$

4) $y = \coth x \rightarrow \frac{dy}{dx} = -\text{csch}^2 x$

5) $y = \text{sech}\, x \rightarrow \frac{dy}{dx} = -\text{sech x} \tanh x$

6) $y = \text{csch}\, x \rightarrow \frac{dy}{dx} = -\text{csch}\, x \coth x$

Example: Find $\frac{dy}{dx}$ of the following functions.

1) $y = \sinh\,(x^2 + 3\sin x + \ln x)$

$$\frac{dy}{dx} = cosh(x^2 + 3\sin x + \ln x).\,(2x + 3\cos x + \frac{1}{x})$$

2) $y = \tanh^{-2}(e^{\tanh^{-1} 2x} + sin\, e^{2x})$

$$\frac{dy}{dx} = -2\ \tanh^{-3}(e^{\tanh^{-1} 2x} + sin\ e^{\ 2x})\,.\,\text{sech}^2(e^{\tan^{-1}\ 2x}$$

$$+ sine^{2x}).\,e^{\tan^{-1}\ 2x}.\left(\frac{2}{1+4x^2}\right) + \cos e^{2x}\,.\,(e^{2x}.\,2)$$

Inverse of hyperbolic function:

1) $y = \sinh^{-1} x \rightarrow x = \sinh y$
2) $y = \cosh^{-1} x \rightarrow x = \cosh y$
3) $y = \tanh^{-1} x \rightarrow x = \tanh y$
4) $y = \coth^{-1} x \rightarrow x = \coth y$
5) $y = \operatorname{sech}^{-1} x \rightarrow x = \operatorname{sech} y$
6) $y = \operatorname{csch}^{-1} x \rightarrow x = \operatorname{csch} y$

Derivative of inverse of hyperbolic function:

1) $y = \sinh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$
2) $y = \cosh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$
3) $y = \tanh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$
4) $y = \coth^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$
5) $y = \operatorname{sech}^{-1} x \rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$
6) $y = \operatorname{csch}^{-1} x \rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1+x^2}}$

Example: Find $\frac{dy}{dx}$ to the following function: $y = \sinh^{-1}(x^2 + \sin^2 x)$

Sol:

$$\frac{dy}{dx} = \frac{2x+2\sin x\cos x}{\sqrt{1+(x^2+\sin^2 x)^2}}$$

Integration

Indefinite Integrals: التكامل الغير محدد

Definition: The set of all antiderivatives of f is the indefinite integral of f with respect to x**,** denoted by:

$$\int f(x)\,dx = F(x) + c$$

The symbol $\int$ *is* an integral sign. The function is f the integrand of the integral, and x is the variable of integration and c is the constant of integral.

Some integration formulas:

1) $\int \frac{du}{dx} dx = u(x) + c$
2) $\int a\,u(x)\,dx = a \int u(x)$, *a is constant.*
3) $\int [u_1(x) + u_2(x) + \cdots]dx = \int u_1(x)dx + \int u_2(x)dx + \ldots$
4) $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + c \;, n \neq -1$

Example: Evaluate the integral $\int (4x^2 + 2x - 1)\,dx$

Sol:

$$\int (4x^2 + 2x - 1)\,dx = 4\int x^2\,dx + 2\int x\,dx - \int dx$$

$$= 4\frac{x^3}{3} + \frac{2x^2}{2} - x + c = \frac{4}{3}x^3 + x^2 - 5x + c$$

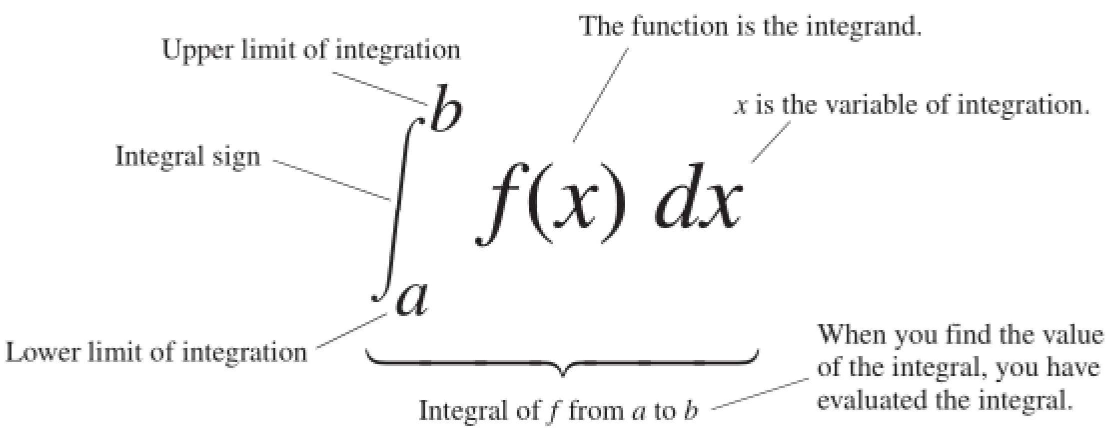
Example: Evaluate the integral $\int (4x - x^2)^2 \, (4 - 2x)dx$

Sol:

$$\int (4x - x^2)^2 \, dx = \frac{(4x-x^2)^3}{3} + c = \frac{1}{3}\,(4x - x^2)^3 + c$$

Definite Integral: التكامل المحدد

The integral $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ over the interval [a, b].



Properties of definite integrals:

If $f(x)$ is a continuous function on [a, b] then:

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2) $\int_a^a dx\,dx = 0$
3) $\int_a^b k\,f(x)\,dx = k\,\int_a^b f(x)\,dx\,, k = conatant.$
4) $\int_a^b [f_1(x) + f_2(x) + f_3(x) + \cdots]dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx + \int_a^b f_3\,dx + \cdots$

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, for\ c\ \in [a,b]$.

The fundamental theorem of integral calculus:

If $f(x)$ is continuous function on [a, b] and $F(x)$ is any solution of $f(x)$ over [a, b], then:

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

Example: Evaluate $\int_{-3}^{2}(6 - x - x^2)\,dx$

Sol:

$$\int_{-3}^{2}(6 - x - x^2)\,dx = 6x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-3}^{2} = \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right) = \frac{125}{6}$$

Example: If $f(x)$ is a continuous, show that: $\int_0^1 f(x)\,dx = \int_0^1 f(1-t)\,dt$

Sol:

$$let\ x = 1 - t \rightarrow dx = -dt$$

$$at\ x = 0 \rightarrow t = 1, x = 1 \rightarrow t = 0$$

$$\int_0^1 f(x)\,dx = \int_1^0 f(1-t)\,.-dt = \int_0^1 f(1-t)\,dt$$

Method of Integration:

a) Integration formula:

1) $\int u^n\,du = \frac{u^{n+1}}{n+1} + c \quad , n \neq -1$

2) $\int \frac{du}{u} = \ln u + c$

3) $\int e^u \, du = e^u + c$

4) $\int a^u \, du = \frac{a^u}{\ln a} + c$

5) $\int \sin u \, du = -\cos u + c$

6) $\int \cos u \, du = \sin u + c$

7) $\int \sec^2 u \, du = \tan u + c$

8) $\int \csc^2 u \, du = -\cot u + c$

9) $\int \sinh u \, du = \cosh u + c$

10) $\int \cosh u \, du = \sinh u + c$

11) $\int \operatorname{sech}^2 u \, du = \tanh u + c$

12) $\int \operatorname{csch}^2 u \, du = -\coth u + c$

13) $\int \sec u \tan u \, du = \sec u + c$

14) $\int \csc u \cot u \, du = -\csc u + c$

Examples:

1) $\int_0^{2\pi} \sin x \, dx = -\cos x|_0^{2\pi} = -(\cos 2\pi - \cos 0) = -(1-1) = 0$

2) $\int \sec 2x \tan 2x \, dx = \frac{1}{2}\sec(2x) + c$

3) $\int 5^{x^3} 15x^2 dx = \frac{5^{x^3}}{\ln 5} + c$

4) $4\int \operatorname{csch}^2(4x)dx = -\coth 4x + c$

b) Integration substitution:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boldsymbol{I = \int f[g(x)]\, \grave{g}(x) dx}$$

- **The steps to evaluate the integral is:**

1- Substitute $u = g(x)$and $du = \grave{g}(x)$ to obtain the integral $\int \boldsymbol{f(u)\, du}$

2- Integrate with respect to u.

3- Replace u by $g(x)$ in the result.

Example: Using Substitution to find $\int \cos{(7\theta + 5)}\, d\theta$

Sol:

$$let\ u = 7\theta + 5 \rightarrow du = 7\ d\theta \qquad \therefore d\theta = \frac{1}{7} du$$

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u * \frac{1}{7}\, du = \frac{1}{7} \int \cos u\, du$$

$$= \frac{1}{7} \sin u + c = \frac{1}{7} \sin(7\theta + 5) + c$$

Example: Using Substitution to find $\int \mathrm{x}^2\ \sin(x^3)\, dx$

Sol:

$$let\ u = x^3 \rightarrow du = 3x^2dx \qquad \therefore dx = \frac{1}{3x^2}du$$

$$\int x^2 \sin(x^3)\,dx = \int \sin(u) * \frac{1}{3}du = \frac{1}{3}\int \sin(u)\,du$$

$$= -\frac{1}{3}\cos(u) + c = \frac{-1}{3}\cos(x^3) + c$$

Example: Using Substitution to find $\frac{\int 2zdz}{\sqrt[3]{z^2+1}}$

Sol:

$$let\ u = z^2 + 1 \rightarrow du = 2z\,dz$$

$$\frac{\int 2zdz}{\sqrt[3]{z^2+1}} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}}\,du = \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}\,u^{2/3} + c = \frac{3}{2}\,(z^2+1)^{\frac{2}{3}} + c$$