Chapter three
Derivatives

Let $y = f(x)$ be a function of x. If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 **– Find the derivative of the function :** $f(x) = \dfrac{1}{\sqrt{2x+3}}$

***Sol.*:**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{\sqrt{2(x+\Delta x)+3}} - \dfrac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

<u>Rules of derivatives</u> : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\dfrac{d}{dx} c = 0$
2. $\dfrac{d}{dx} u^n = n u^{n-1} \dfrac{du}{dx} \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{u}\right) = -\dfrac{1}{u^2}\dfrac{du}{dx}$
3. $\dfrac{d}{dx} cu = c\dfrac{du}{dx}$
4. $\dfrac{d}{dx}(u \mp v) = \dfrac{du}{dx} \mp \dfrac{dv}{dx}$; $\dfrac{d}{dx}(u \mp v \mp w) = \dfrac{du}{dx} \mp \dfrac{dv}{dx} \mp \dfrac{dw}{dx}$
5. $\dfrac{d}{dx}(u.v) = u.\dfrac{dv}{dx} + v\dfrac{du}{dx}$

and $$\frac{d}{dx}(u.v.w) = u.v\frac{dw}{dx} + u.w\frac{dv}{dx} + v.w\frac{du}{dx}$$

6. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad where \quad v \neq 0$$

_EX-2_- Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = (x^2+1)^5$ $\qquad b)\ y = [(5-x)(4-2x)]^2$

$c)\ y = (2x^3 - 3x^2 + 6x)^{-5}$ $\qquad d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y = \frac{(x^2+x)(x^2-x+1)}{x^3}$ $\qquad f)\ y = \frac{x^2-1}{x^2+x-2}$

Sol.-

$a)\quad \frac{dy}{dx} = 5(x^2+1)^4.2x = 10x(x^2+1)^4$

$b)\quad \frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$\qquad = 8(5-x)(2-x)(2x-7)$

$c)\quad \frac{dy}{dx} = -5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$\qquad = -30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\quad y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$

$\qquad \Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$

$e)\quad y = \frac{(x+1)(x^2-x+1)}{x^3} \Rightarrow$

$$\frac{dy}{dx} = \frac{x^3[(x^2-x+1)+(x+1)(2x-1)] - 3x^2(x+1)(x^2-x+1)}{x^6} = -\frac{3}{x^4}$$

$f)\quad \frac{dy}{dx} = \frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2} = \frac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule:

1. **Suppose that $h = g_o f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x , then y is a differentiable function of x :**

$$y = g(t) \ \ and \ \ t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 **– Use the chain rule to express dy / dx in terms of x and y :**

$a)\quad y = \frac{t^2}{t^2+1} \qquad and \quad t = \sqrt{2x+1}$

$b)\quad y = \frac{1}{t^2+1} \qquad and \quad x = \sqrt{4t+1}$

$c)\quad y = \left(\frac{t-1}{t+1}\right)^2 \qquad and \quad x = \frac{1}{t^2} - 1 \quad at \quad t = 2$

$d)\quad y = 1 - \frac{1}{t} \qquad and \quad t = \frac{1}{1-x} \quad at \quad x = 2$

Sol.-

$$a)\quad y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where}\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where}\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow\frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\qquad \text{at}\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

Higher derivatives : **If a function *y = f(x)* possesses a derivative at every point of some interval , we may form the function *f '(x)* and talk**

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y'' , f''(x) , or \ \frac{d^2f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)}, f^{(n)}(x), \frac{d^ny}{dx^n} .$$

***EX-4*- Find all derivatives of the following function :**

$$y = 3x^3 - 4x^2 + 7x + 10$$

***Sol.*-**

$$\frac{dy}{dx}=9x^2-8x+7 \quad , \quad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \quad , \quad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

***Ex-5* – Find the third derivative of the following function :**

$$y=\frac{1}{x}+\sqrt{x^3}$$

***Sol.*-**

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

***Implicit Differentiation*:** **If the formula for *f* is an algebraic combination of powers of *x* and *y*. To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to *x* .**

***EX-6*- Find $\frac{dy}{dx}$ for the following functions:**

$a)\ x^2.y^2 = x^2 + y^2$

$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \frac{x-y}{x-2y} = 2 \text{ at } P(3,1)$

$d)\ xy + 2x - 5y = 2 \text{ at } P(3,2)$

Sol.

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

***Exponential functions* :** **If *u* is any differentiable function of *x* , then :**

$$7)\quad \frac{d}{dx}a^u = a^u.\ln a.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u.\frac{du}{dx}$$

EX-7 –Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = 2^{3x}$ $\qquad$ $b)\ y = 2^x.3^x$

$c)\ y = (2^x)^2$ $\qquad$ $d)\ y = x.2^{x^2}$

$e)\ y = e^{(x+e^{5x})}$ $\qquad$ $f)\ y = e^{\sqrt{1+5x^2}}$

Sol.-

$$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3\ln 2$$

$$b)\ y = 2^x.3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x.\ln 6$$

$$c)\ y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}\ln 2.2 = 2^{2x+1}\ln 2$$

$$d)\ y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2}\ln 2.2x + 2^{x^2} = 2^{x^2}(2x^2\ln 2 + 1)$$

$$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1 + 5e^{5x})$$

$$f)\ y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\frac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\frac{5x}{\sqrt{1+5x^2}}$$

Logarithm functions *:* **If u is any differentiable function of x , then :**

$$8)\ \frac{d}{dx}\log_a u = \frac{1}{u.\ln a}.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

EX-8 – Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = \log_{10} e^x$ $\qquad$ $b)\ y = \log_5(x+1)^2$

$c)\ y = \log_2(3x^2+1)^3$ $\qquad$ $d)\ y = \left[\ln(x^2+2)^2\right]^3$

$e)\ y + \ln(xy) = 1$ $\qquad$ $f)\ y = \frac{(2x^3-4)^{\frac{2}{3}}.(2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

Sol. –

$$a)\ y=\log_{10}e^{x} \Rightarrow y=x\log_{10}e \Rightarrow \frac{dy}{dx}=\log_{10}e=\frac{\ln e}{\ln 10}=\frac{1}{\ln 10}$$

$$b)\ y=\log_{5}(x+1)^{2}=2\log_{5}(x+1) \Rightarrow \frac{dy}{dx}=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=3\log_{2}(3x^{2}+1) \Rightarrow \frac{dy}{dx}=\frac{3}{3x^{2}+1}\cdot\frac{6x}{\ln 2}=\frac{18x}{(3x^{2}+1)\ln 2}$$

$$d)\ \frac{dy}{dx}=3\left[2\ln(x^{2}+2)\right]^{2}\frac{2}{x^{2}+2}.2x=\frac{48x\left[\ln(x^{2}+2)\right]^{2}}{x^{2}+2}$$

$$e)\ y+\ln x+\ln y=1 \Rightarrow \frac{dy}{dx}+\frac{1}{x}+\frac{1}{y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x(y+1)}$$

$$f)\ \ln y=\frac{2}{3}\ln(2x^{3}-4)+\frac{5}{2}\ln(2x^{2}+3)-2\ln(7x^{3}+4x-3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{3}.\frac{6x^{2}}{2x^{3}-4}+\frac{5}{2}.\frac{4x}{2x^{2}+3}-2.\frac{21x^{2}+4}{7x^{3}+4x-3}$$

$$\Rightarrow \frac{dy}{dx}=2y\left[\frac{2x^{2}}{2x^{3}-4}+\frac{5x}{2x^{2}+3}-\frac{21x^{2}+4}{7x^{3}+4x-3}\right]$$

<u>Trigonometric functions</u> **: If *u* is any differentiable function of *x* , then :**

$$9)\ \frac{d}{dx}\sin u=\cos u.\frac{du}{dx}$$

$$10)\ \frac{d}{dx}\cos u=-\sin u.\frac{du}{dx}$$

$$11)\ \frac{d}{dx}\tan u=\sec^{2}u.\frac{du}{dx}$$

$$12)\ \frac{d}{dx}\cot u=-\csc^{2}u.\frac{du}{dx}$$

$$13)\ \frac{d}{dx}\sec u=\sec u.\tan u.\frac{du}{dx}$$

$$14)\ \frac{d}{dx}\csc u=-\csc u.\cot u.\frac{du}{dx}$$

***<u>EX-9</u>*-** **Find** $\frac{dy}{dx}$ **for the following functions :**

$a)\ y = \tan(3x^2)$ $\qquad$ $b)\ y = (\csc x + \cot x)^2$

$c)\ y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$ $\qquad$ $d)\ y = \tan^2(\cos x)$

$e)\ x + \tan(xy) = 0$ $\qquad$ $f)\ y = \sec^4 x - \tan^4 x$

Sol.-

$$a)\ \frac{dy}{dx} = \sec^2(3x^2).6x = 6x.\sec^2(3x^2)$$

$$b)\ \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x.\cot x - \csc^2 x) = -2\csc x.(\csc x + \cot x)^2$$

$$c)\ \frac{dy}{dx} = 2\cos\frac{x}{2}.\frac{1}{2} - \left[x(-\sin\frac{x}{2}).\frac{1}{2} + \cos\frac{x}{2}\right] = \frac{x}{2}.\sin\frac{x}{2}$$

$$d)\ \frac{dy}{dx} = 2.\tan(\cos x).\sec^2(\cos x).(-\sin x) = -2.\sin x.\tan(\cos x).\sec^2(\cos x)$$

$$e)\ 1 + \sec^2(xy).(x\frac{dy}{dx} + y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y.\sec^2(xy)}{x.\sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$$

$$f)\ \frac{dy}{dx} = 4\sec^3 x.\sec x.\tan x - 4.\tan^3 x.\sec^2 x = 4\tan x.\sec^2 x$$

EX-10- **Prove that :**

$a)\ \frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$ $\qquad$ $b)\ \frac{d}{dx}\sec u = \sec u.\tan u.\frac{du}{dx}$

Proof :

$$a)\quad L.H.S. = \frac{d}{dx}\tan u = \frac{d}{dx}\frac{\sin u}{\cos u} = \frac{\cos u.\cos u.\frac{du}{dx} - \sin u.(-\sin u)\frac{du}{dx}}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}.\frac{du}{dx} = \frac{1}{\cos^2 u}.\frac{du}{dx} = \sec^2 u.\frac{du}{dx} = R.H.S.$$

$$b)\quad L.H.S. = \frac{d}{dx}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{1}{\cos u}.\frac{\sin u}{\cos u}.\frac{du}{dx} = \sec u.\tan u.\frac{du}{dx} = R.H.S.$$

The inverse trigonometric functions : **If *u* is any differentiable function**