Laplace transform of t^n.

$$\rightarrow \mathcal{L}[f(t)] = F(s) \doteq \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$\mathcal{L}(t^n) = \int_0^\infty e^{-st} \cdot t^n\, dt$$

S * t = u

Diff. with respect to u

S dt/du = du/du

S dt/du = 1

where,

t=0 then u=0

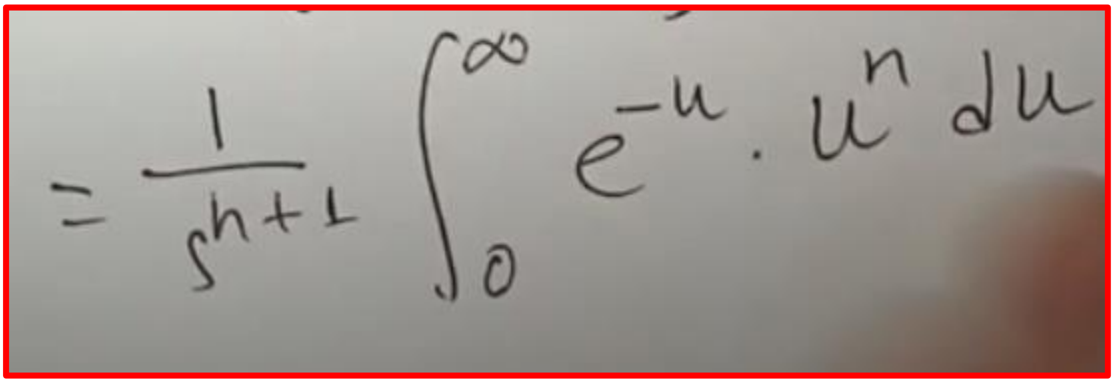
t=∞ then u=∞

then,

t= u/s

$$\mathcal{L}(t^n) = \int_0^\infty e^{-u} \cdot \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \cdot \frac{u^n}{s^{n+1}}\, du$$



The gamma function is:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} \, dx = n!$$

Which is similar to?

$$\frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du \quad \textit{The blue part is equal to } \Gamma(n+1) = n!$$

Then,

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$\mathcal{L}\{\sin at\}$	$=$	$\mathcal{L}\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$	Sine Exponential Formulation
	$=$	$\frac{1}{2i}\left(\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}\right)$	Linear Combination of Laplace Transforms
	$=$	$\frac{1}{2i}\left(\frac{1}{s - ia} - \frac{1}{s + ia}\right)$	Laplace Transform of Exponential
	$=$	$\frac{1}{2i}\left(\frac{s + ia - s + ia}{s^2 + a^2}\right)$	simplifying
	$=$	$\frac{1}{2i}\left(\frac{2ia}{s^2 + a^2}\right)$	simplifying
	$=$	$\frac{a}{s^2 + a^2}$	simplifying

$$\begin{aligned} \mathcal{L}\left\{e^{iat}\right\} &= \frac{1}{s - ia} && \text{Laplace Transform of Exponential} \\ &= \frac{s + ia}{s^2 + a^2} && \text{multiply top and bottom by } s + ia \end{aligned}$$

lso:

$$\begin{aligned} \mathcal{L}\left\{e^{iat}\right\} &= \mathcal{L}\left\{\cos at + i \sin at\right\} && \text{Euler's Formula} \\ &= \mathcal{L}\left\{\cos at\right\} + i\mathcal{L}\left\{\sin at\right\} && \text{Linear Combination of Laplace Transforms} \end{aligned}$$

o:

$$\begin{aligned} \mathcal{L}\left\{\cos at\right\} &= \mathfrak{Re}\left(\mathcal{L}\left\{e^{iat}\right\}\right) \\ &= \mathfrak{Re}\left(\frac{s + ia}{s^2 + a^2}\right) \\ &= \frac{s}{s^2 + a^2} \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left\{\cos at\right\} &= \mathcal{L}\left\{\frac{e^{iat} + e^{-iat}}{2}\right\} && \text{Cosine Exponential Formulation} \\ &= \frac{1}{2}\left(\mathcal{L}\left\{e^{iat}\right\} + \mathcal{L}\left\{e^{-iat}\right\}\right) && \text{Linear Combination of Laplace Transforms} \\ &= \frac{1}{2}\left(\frac{1}{s - ia} + \frac{1}{s + ia}\right) && \text{Laplace Transform of Exponential} \\ &= \frac{1}{2}\left(\frac{s + ia + s - ia}{s^2 + a^2}\right) && \text{simplifying} \\ &= \frac{s}{s^2 + a^2} && \text{simplifying} \end{aligned}$$

First Shifting Property:

If $Lf(t) = F(s), whens > a$ then,

In words, the substitution s−a for s in the transform corresponds to the multiplication of the original function by e^{at}.

$F(s) = \int_0^\infty e^{-st} f(t) dt$

$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) dt$

$F(s-a) = \int_0^\infty e^{-st+at} f(t) dt$

$F(s-a) = \int_0^\infty e^{-st} e^{at} f(t) dt$

$F(s-a) = \mathcal{L}\left\{e^{at} f(t)\right\}$

Proof of Second Shifting Property

$$g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt$$

$$\mathcal{L}\{g(t)\} = \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$\mathcal{L}\{g(t)\} = \int_a^\infty e^{-st} f(t-a)\, dt$$

Let

$z = t - a$

$t = z + a$

$dt = dz$

when $t = a,\ z = 0$

when $t = \infty,\ z = \infty$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(z+a)} f(z)\, dz$$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-sz-sa} f(z)\, dz$$

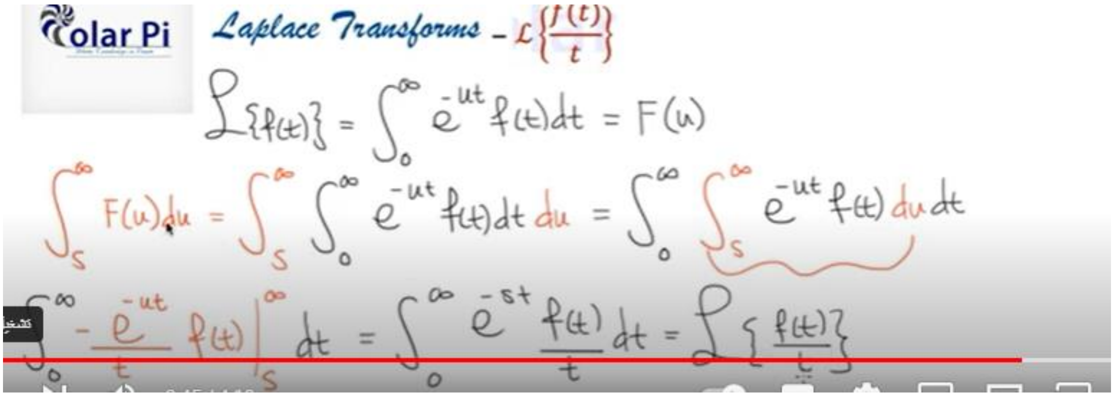
$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-sz} e^{-sa} f(z)\, dz$$

$$\mathcal{L}\{g(t)\} = e^{-sa} \int_0^\infty e^{-sz} f(z)\, dz$$

$$\mathcal{L}\{g(t)\} = e^{-as} \mathcal{L}\{f(z)\}$$

$$\mathcal{L}\{g(t)\} = e^{-as} \mathcal{L}\{f(t-a)\}$$

Then Laplace of $g(t) = e^{-as} F(s)$



Laplace Transformations of Derivatives and integrals

Suppose that f(t) is continuous for all t≥ 0, and has derivative f' (t) which is continues on every finite interval in the range t≥ 0. Then, the L.T. of the derivative f' (t) exists:

If F(s) = $\mathcal{L}[f(t)]$

Then $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Proof: $\mathcal{L}[f'(t)]=\int_0^{\infty} f'(t).\ e^{-st}\ dt$

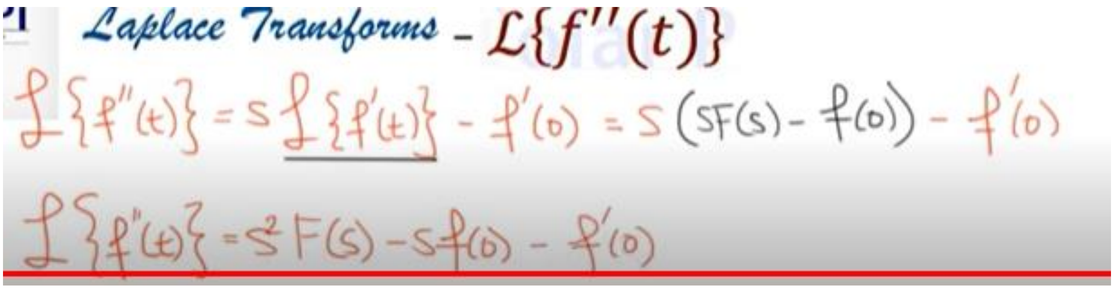
Integrating by parts ($\int udv = uv - \int vdu$)

dv=f' (t) dt & u=e^{-st}

v=f(t) & du= -s . e^{-st} dt

$\mathcal{L}[f'(t)] = [e^{-st}\ [f(t)]_0^{\infty} + s\int_0^{\infty} e^{-st} f(t)dt$

$\therefore\ \mathcal{L}[f'(t)] = sF(s) - f(0)$



$$\mathcal{L}\{-t\,f(t)\} = F'(s) \qquad \mathcal{L}\{f(t)\} = F(s)$$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty f(t)\,\frac{d}{ds}\left(e^{-st}\right)dt$$

$$= \int_0^\infty f(t)\left(-t\,e^{-st}\right)dt$$

$$= \int_0^\infty f(t) \frac{d}{ds}\left(e^{-st}\right) dt$$

$$= \int_0^\infty f(t)\left(-t\, e^{-st}\right) dt$$

$$F'(s) = \int_0^\infty e^{-st}\left(-t\, f(t)\right) dt$$

$$F'(s) = \mathcal{L}\{-t\, f(t)\}$$

$$\mathcal{L}[f'''(t)] = s^3F(s) - s^2\, F(0) - sf'(0) - f''(0)$$

Theorem 1: Linearity of the Laplace transformation

For any function f(t) and g(t) whose laplace transform exist and any constant a &b, we have:

$$\mathcal{L}[af(t) \mp bg(t)] = a\mathcal{L}[f(t)] \mp b\mathcal{L}[g(t)]$$

Proof: $\mathcal{L}[af(t) \mp bg(t)]$=$\int_0^\infty e^{-st}\, [af(t) \mp bg(t)]dt$

= $a\int_0^\infty e^{-st}\; f(t)dt \;+ b\int_0^\infty e^{-st}\; g(t)dt$

$= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

Theorem 2: a first shifting theorem

Let $\mathcal{L}[f(t)]$=F(s), then $\mathcal{L}[e^{at}\, f(t)] = F(s-a)$

Proof: $F(s) = \int_0^\infty e^{-st} . f(t)dt$

So, $\mathcal{L}[e^{at}\, f(t)]$**=** $\int_0^\infty f(t)\, e^{at}\; .e^{-st}\, dt$ **=**$\int_0^\infty f(t)\, e^{-(s-a)t}\; dt$ **=F(s-a)**

i.e. the multiplication of f(t) by (e^{at}) shifts the variable (s) in the L.T. to (s-a).

Theorem 3: a second shifting theorem

If $\mathcal{L}[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a) \ldots\ldots t > a \\ 0 \ldots\ldots t < a \end{cases}$

i.e $\boxed{\mathcal{L}[g(t)] = e^{-as} F(s)}$

Proof: $\mathcal{L}[g(t)] = \int_0^\infty f(t-a)\ e^{-st}\, dt$ changing the variable in the integral to (t-a)=τ, dt=d τ

G(s)= $\int_0^\infty f(\tau)\ e^{-s(a+\tau)}\, dt$ $= e^{-as} \int_0^\infty e^{-s\tau}.f(\tau)dt = e^{-as} F(s)$

Theorem 4: Change of scale

If F(s) = $\mathcal{L}[f(t)]$

$\therefore \mathcal{L}[f(at)] = \frac{1}{a}$ F($\frac{s}{a}$)

Proof: $\mathcal{L}[f(at)] = \int_0^\infty f(at)\ e^{-st}\, dt$

Let at=u $\Rightarrow a.dt = du$

$\mathcal{L}[f(at)] = \int_0^\infty f(u)\ e^{-s\frac{u}{a}}\ .\frac{1}{a}$.du=$\frac{1}{a}\int_0^\infty f(u)\ e^{-\frac{su}{a}}.du$

$=\frac{1}{a}$ F(s/a).

Laplace transform of integral

If F(s)= $\mathcal{L}$[f(t)]

then, $\mathcal{L}\left[\int_0^\infty f(t)\ dt\right] = \int_0^\infty [f(t)\, dt].\ e^{-st}\ dt$

let u= $f(t)\, dt$ $\quad\quad dv = e^{-st}dt$

du=f(t) dt $\quad\quad v = -\frac{1}{s}e^{-st}$

$$\mathcal{L}\left[\int_0^t f(t)\ dt\right] = \{[\int_0^t f(t)dt]\left[-\frac{1}{s}e^{-st}\right]\}_0^\infty - \int -\frac{1}{s}e^{-st}f(t)dt$$

$= -\frac{1}{s} e^{-st} \int_0^t f(t)\ dt]_0^\infty + \frac{1}{s} . F(s)$

Since $e^{-st} \Longrightarrow 0\ \ as\ t \Longrightarrow \infty$

& t$\Longrightarrow$ 0, the integral in this term vanishes

$$\therefore \mathcal{L}\left[\int_0^t f(t)\ dt\right] = \frac{F(s)}{s}$$