Mathematics 1

المرحلة الأولى / المحاضرة الرابعة

قسم الأنظمة الطبية الذكية

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رياضيات عملي

Applications of Derivatives

Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem.*

Example 1: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution: Figure 1 shows a partially filled conical tank. The variables in the problem are:

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

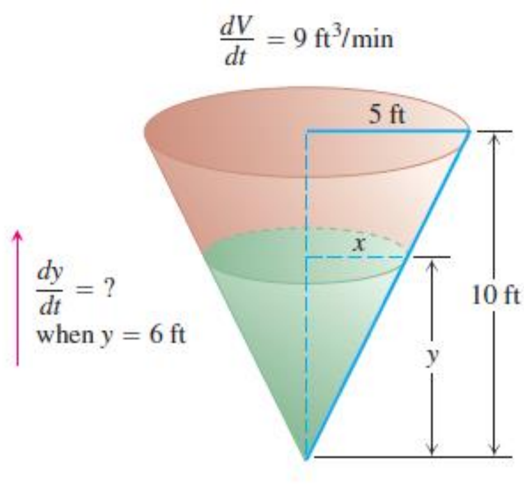

Figure 1

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \quad \text{and } dV/dt = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 1 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \text{ or } x = \frac{y}{2}$$

Therefore, find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

To give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} . 3y^2 \frac{dy}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft /min.

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).

4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

Example 2: A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps:

1. *Draw a picture and name the variables and constants* (Figure 2). The variables in the picture are:

θ = the angle in radians the range finder makes with the ground.

y = the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

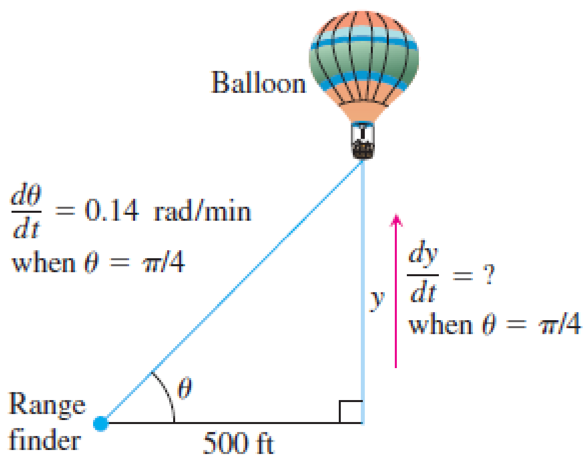

Figure 2

2. *Write down the additional numerical information.*

$$d\theta/dt = 0.14 \text{ rad/min} \quad \text{when } \theta = \pi/4$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$
4. *Write an equation that relates the variables y and* θ

$$y/500 = \tan\theta \quad \text{or } y = 500 \tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how (which we want) is related to (which we know).

$$dy/dt = 500\,(\sec^2\theta)\, d\theta/dt$$

6. *Evaluate with* $\theta = \pi/4$ *and* $d\theta/dt = 0.14$ *to find* dy/dt.

$$dy/dt = 500\,(\sqrt{2})^2(0.14) = 140 \quad [\sec \pi/4 = \sqrt{2}]$$

At the moment in question, the balloon is rising at the rate of 140 ft/min.

Example 3: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution: We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3).
We let t represent time and set

x = position of car at time t
y = position of cruiser at time t
s = distance between car and cruiser at time t .

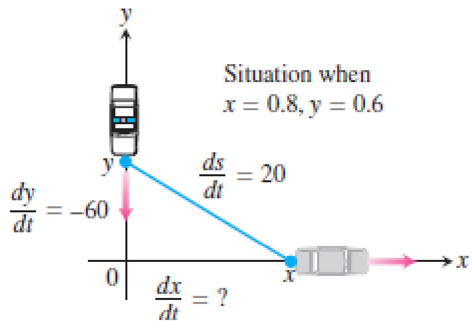

Figure 3

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$x = 0.8$ mi, $y = 0.6$ mi, $dy/dt = -60$ mph, $ds/dt = 20$ mph.

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car’s speed is 70 mph.

Example 4: A water trough is 10 m long and a cross section has the shape of isosceles trapezoid as shown in Figure 4. If the trough is being filled with water at the rate of 0.2 m^3/ min, how fast is the water level rising when the water is 30 cm deep?

Solution:

V = volume of water

$V = \frac{(0.3+2x)+0.3}{2} * h * 10$ = (0.6 + 2 *x*) * 5*h* = 3*h* + 10*xh*

From similarity of triangles (Figure 4):

$\frac{x}{h} = \frac{0.25}{0.5} = \frac{1}{2}$

$x = h/2$

$V = 3\frac{dh}{dt} + 10\, h\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dv/dt}{3+10h}$$

At h = 30 cm = 0.3 m $\quad \frac{dh}{dt} = \frac{0.2}{3+10(0.3)} = \frac{0.2}{6} = \frac{1}{30}$ *m/min*

Example 5: A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of in order to keep the aircraft within its direct line of sight?

Solution: The aircraft *A* and radar station *R* are pictured in the coordinate plane, using the positive *x*-axis as the horizontal distance at sea level from *R* to *A*, and the positive *y*-axis as the vertical altitude above sea level. We let *t* represent time and observe that *y* = 12000 is a constant. The general situation and line-of-sight angle θ are depicted in Figure 5. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

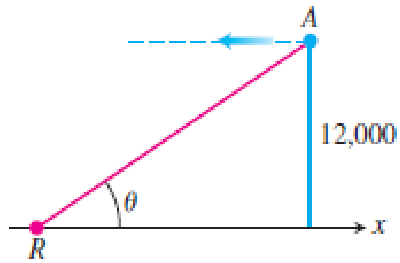

Figure 5

From Figure 5, we see that

$$12{,}000/\,x = \tan\theta \quad \text{or } x = 12{,}000 \cot\theta$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280} \cot\theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -\frac{12{,}00}{528} \csc^2\theta \frac{d\theta}{dt}$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$,so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3\ deg/sec$ to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600) \text{ rad/hr} \qquad [\text{ 1 hr} = 3600 \text{ sec, } 1 \text{ deg} = \pi/180 \text{ rad}]$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar.

Example 6: Figure 6 (a) shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking rapidly away from the vertical line *PW* at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from *PW*?

Solution: We let *OM* be the horizontal line of length *x* ft from a point *O* directly below the pulley to the worker's hand *M* at any instant of time (Figure 6). Let *h* be the height of the weight *W* above *O*, and let z denote the length of rope from the pulley *P* to the worker's hand. We want to know *dh/dt* when $x = 21$ given that *dx/dt* = 6.

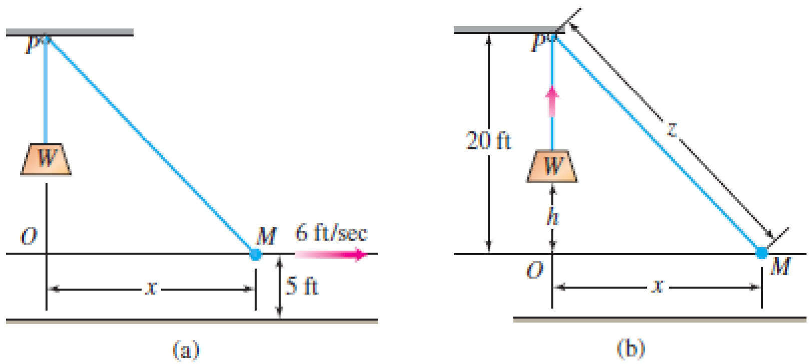

Figure 6

Note that the height of *P* above *O* is 20 ft because *O* is 5 ft above the ground. We assume the angle at *O* is a right angle.

At any instant of time *t* we have the following relationships (see Figure 5b):

$$20 - h + z = 45 \quad \text{[Total length of rope is 45 ft]}$$

$$20^2 + x^2 = z^2 \text{ [Angle at } O \text{ is a right angle]}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2 \quad \ldots\ldots (1)$$

Differentiating both sides with respect to *t* gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt}$$

and solving this last equation for *dh/dt* we find

$$\frac{dh}{dt} = \left(\frac{x}{25+h}\right)\frac{dx}{dt} \quad \ldots\ldots\ldots (2)$$

Since we know *dx/dt*, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

So that

$$(25 + h)^2 = 841 \quad \text{or} \quad 25 + h = 29$$

Equation (2) now gives

$$dh/dt = (21/29)*6 \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft