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المرحلة : الثانية
السنة الدراسية : 2023-2024

عنوان المحاضرة: Complex Numbers الاعداد المركبة

Polar Form الصيغة القطبية

Euler's Equation معادلة أويلر

Complex Numbers

A Complex number can be represented by an expression of the form $(a+ib)$ where:

a and b :- are real numbers.

i :- is a symbol with the property that $i^2=-1$ i.e $(i=\sqrt{-1})$ is called imaginary unit.

Definitions:-

(1) $Z=a+ib$ where Z: is a complex number
a: real part
b: imaginary part

$\bar{Z}=a-ib$ where $\bar{Z}$: is a complex conjugate to the complex number (Z)

Ex:- If $Z=3+4i$ then $\bar{Z}=3-4i$

(2) The absolute value or modulus of $(a+ib)$ is defined as

$$|a+ib|=\sqrt{a^2+b^2}$$

(3) If $Z_1=a+ib$ and $Z_2=c+id$ are equal then $a=c$ and $b=d$

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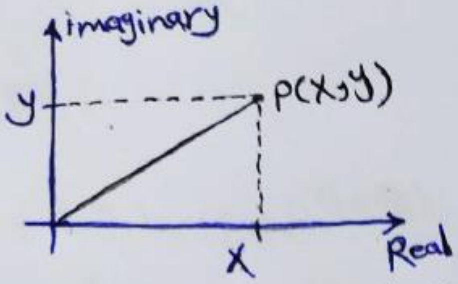
Complex numbers representation :-

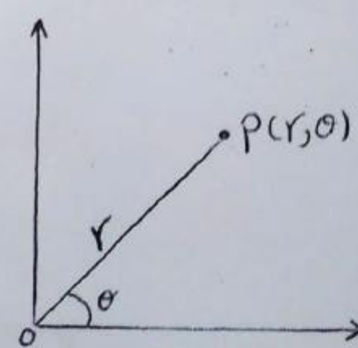
There are two geometric representations of the complex number $(z = x + iy)$

(1) The point $P(x,y)$ in xy-plane which is called Argand diagram or complex plane. where x-axis represents the real axis and y-axis represents the imaginary axis.



(2) The vector $\overrightarrow{OP}$ from the origin to $P(r,\theta)$ which is called polar form of complex numbers, where r and θ are called polar Coordinates.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Where θ is the amplitude or argument angle with x-axis.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \leftrightarrow \text{Euler's formula.}$$

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Properties of complex number :-

If $z_1 = x_1 + i y_1 = r_1 e^{i\theta_1}$

$z_2 = x_2 + i y_2 = r_2 e^{i\theta_2}$ then

(1) $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

(2) $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

(3) $K \cdot z_1 = K(x_1 + i y_1) = Kx_1 + Ky_1 i$

(4) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

(5) $z_1 \cdot \bar{z}_1 = |z_1|^2$ and $z_2 \cdot \bar{z}_2 = |z_2|^2$

(6) $\dfrac{z_1}{z_2} = \dfrac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \dfrac{(x_1 + i y_1)(x_2 - i y_2)}{(x_2 + i y_2)(x_2 - i y_2)} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

$= \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

(7) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{r_1}{r_2}$ $\quad r_2 \neq 0$

(8) $\bar{\bar{z}}_1 = z_1$ and $\bar{\bar{z}}_2 = z_2$

(9) $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$

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EX:- Let $Z_1 = 2+3i$ and $Z_2 = 4+i$ find

① $Z_1 + Z_2$ ② $Z_1 - Z_2$

③ $Z_1 \cdot Z_2$ ④ Z_1 / Z_2

Solution:-

① $Z_1 + Z_2 = (2+3i) + (4+i) = \boxed{6+4i}$

② $Z_1 - Z_2 = (2+3i) - (4+i) = \boxed{-2-2i}$

③ $Z_1 \cdot Z_2 = (2+3i) * (4+i) = 8+2i+12i+3i^2$
$= 8+14i-3 = \boxed{5+14i}$

$i^2 = -1$

④ $\dfrac{Z_1}{Z_2} = \dfrac{(2+3i)}{(4+i)} * \dfrac{(4-i)}{(4-i)} = \dfrac{8-2i+12i-3i^2}{(4)^2+(1)^2}$

$= \dfrac{8+10i+3}{17} = \boxed{\dfrac{11+10i}{17}}$

EX:- Put the complex number $(1-i\sqrt{3})$ in the polar form

Solution:-

$Z = 1 - i\sqrt{3} = x + iy$

$r = |Z| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = \boxed{2}$

$\theta = \tan^{-1} \dfrac{y}{x} = \tan^{-1} \dfrac{-\sqrt{3}}{1} = \dfrac{-\pi}{3} = \boxed{-60}$

$Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$= 2(\cos(\frac{-\pi}{3}) + i\sin(\frac{-\pi}{3})) = 2e^{-i\frac{\pi}{3}}$

$= 2(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = 2e^{-i\frac{\pi}{3}}$

Note:- $\ln re^{i\theta} = \ln r + i\theta$

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Ex:- Find $\ln(-2)$

Solution:-

$\ln(re^{i\theta}) = \ln r + i\theta$

$\ln(-2) = \ln(-2 + i0)$

$r = \sqrt{(-2)^2 + (0)^2} = \boxed{2}$

$\theta = \tan^{-1}\frac{0}{-2} \Rightarrow \boxed{\theta = \pi}$

$\therefore \ln(-2) = \ln 2 + i\pi$

Ex:- Find x, y if $(3+4i)^2 - 2(x - iy) = x + iy$

Solution:-

$(3+4i)^2 - 2x + 2iy = x + iy$

$9 + 24i + 16i^2 - 2x + i2y = x + iy$

$i^2 = -1 \Rightarrow 9 + 24i - 16 - 2x + i2y = x + iy$

$-7 - 2x = x \Rightarrow -7 = x + 2x$

$3x = -7 \Rightarrow \boxed{x = -7/3}$

$24i + i2y = iy \quad] \div i$

$24 = y - 2y \Rightarrow -y = 24$

$\therefore \boxed{y = -24}$

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De Moivre's Theorem :-

If $Z = r(\cos\theta + i\sin\theta)$ and (n) is a positive integer number then:

$$Z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$Z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$

$$= r^{1/n}\left[\cos\left(\frac{\theta + 2\pi K}{n}\right) + i\sin\left(\frac{\theta + 2\pi K}{n}\right)\right]$$

where $K = 1, 2, 3, \ldots, n-1$

EX:- Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$

Solution :-

$$r = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \boxed{\frac{1}{\sqrt{2}}}$$

$$\theta = \tan^{-1}\frac{1/2}{1/2} \Rightarrow \theta = 45^\circ = \boxed{\frac{\pi}{4}}$$

$$Z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\therefore \left(\frac{1}{2} + i\frac{1}{2}\right)^{10} = \left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{1}{(\sqrt{2})^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

$$= \frac{1}{32}(0 + i)$$

$$= \boxed{\frac{1}{32}i}$$

Ex:- Find the value of $(-1+i)^{1/3}$.

Solution:-

$r=\sqrt{(-1)^2+(1)^2} = \boxed{\sqrt{2}}$

$\theta = \tan^{-1}\frac{1}{-1} = \boxed{-45^\circ}$

$\theta = 180 - 45 = \boxed{135^\circ}$

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2\pi K}{n}\right) + i\sin\left(\frac{\theta+2\pi K}{n}\right)\right]$$

$$(-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{135+2\pi K}{3}\right) + i\sin\left(\frac{135+2\pi K}{3}\right)\right]$$

$K = 0, 1, 2$ $\qquad (n=3 \Rightarrow n-1=2)$

at $K=0 \Rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{135}{3}\right) + i\sin\left(\frac{135}{3}\right)\right]$

$$= (\sqrt{2})^{1/3}(\cos 45 + i\sin 45)$$

at $K=1 \Rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{135+2\pi}{3}\right) + i\sin\left(\frac{135+2\pi}{3}\right)\right]$

$$= (\sqrt{2})^{1/3}\left[\cos 165 + i\sin 165\right]$$

at $K=2 \Rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{135+4\pi}{3}\right) + i\sin\left(\frac{135+4\pi}{3}\right)\right]$

$$= (\sqrt{2})^{1/3}\left[\cos 285 + i\sin 285\right]$$

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Roots of Equations :

If $Z = re^{i\theta}$ is a complex number different from zero and (n), then there are precisely (n) different complex numbers $w_0, w_1, \dots, w_{n-1}$, that are $(n\text{th})$ roots of Z.

Let $w = \rho e^{i\alpha}$ is $(n\text{th})$ root of $Z = re^{i\theta}$

So $\quad w^n = Z \Rightarrow \rho^n e^{in\alpha} = re^{i\theta}$

$\therefore \quad \rho = \sqrt[n]{r}$ is the real positive $(n\text{th})$ root of $(r$

$$n\alpha \neq \theta$$

We can say they may differ only by an integer multiple of (2π) that is

$$n\alpha = \theta + 2\pi K$$

Where $K = 0, \mp 1, \mp 2, \dots$

$$\alpha = \frac{\theta}{n} + \frac{2\pi K}{n}$$

The $(n\text{th})$ roots of $Z = re^{i\theta}$ are given by

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} \cdot e^{i\left(\frac{\theta}{n} + \frac{2\pi K}{n}\right)}$$

Where $K = 0, \mp 1, \mp 2, \dots, n-1$

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EX:- Find the sixth roots of $Z=-8$

Solution:-

$$Z=-8+0i \Rightarrow r=\sqrt{(-8)^2+(0)^2}=\boxed{8}$$

$$\theta=\tan^{-1}\frac{0}{8} \Rightarrow \boxed{\theta=\pi}$$

$$Z=re^{i\theta} \Rightarrow \boxed{Z=8e^{i\pi}}$$

$$\omega_k=\sqrt[n]{r}\, e^{i\left(\frac{\theta+2\pi k}{n}\right)}=\sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

$$n=6 \Rightarrow k=0,1,2,3,4,5$$

at $\underline{k=0} \Rightarrow \omega_0=\sqrt[6]{8}\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]=\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

at $\underline{k=1} \Rightarrow \omega_1=\sqrt[6]{8}\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]=\sqrt{2}\,i$

at $\underline{k=2} \Rightarrow \omega_2=\sqrt[6]{8}\left[\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right]=\sqrt{2}\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right)$

at $\underline{k=3} \Rightarrow \omega_3=\sqrt[6]{8}\left[\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right]=\sqrt{2}\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)$

at $\underline{k=4} \Rightarrow \omega_4=\sqrt[6]{8}\left[\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}\right]=-\sqrt{2}\,i$

at $\underline{k=5} \Rightarrow \omega_5=\sqrt[6]{8}\left[\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right]=\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right).$

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Ex:- Find the roots of the equation $Z^3 = -1 + i$

Solution:-

$$-1 + i = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{(-1)^2 + (1)^2} = \boxed{\sqrt{2}}$$

$$\theta = \tan^{-1}\frac{1}{-1} \Rightarrow \theta = \boxed{\frac{3\pi}{4}}$$

$$Z^3 = -1 + i \Rightarrow Z = \sqrt[3]{-1+i} = (-1+i)^{1/3}$$

$$\therefore (-1+i)^{1/3} = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + 2\pi k\right) + i\sin\frac{3\pi}{4} + 2\pi k)\right]^{1/3}$$

$$= (\sqrt{2})^{1/3}\left[\cos\left(\frac{3\pi}{4n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{3\pi}{4n} + \frac{2\pi k}{n}\right)\right]$$

$K = 0, 1, 2$ & $n = 3$

at $\underline{K=0} \Rightarrow Z_0 = (\sqrt{2})^{1/3}\left[\cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}\right] = (\sqrt{2})^{1/3}(1+i)$

at $\underline{K=1} \Rightarrow Z_1 = (\sqrt{2})^{1/3}\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right]$

at $\underline{K=2} \Rightarrow Z_2 = (\sqrt{2})^{1/3}\left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right]$

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H.W

Complex Numbers

1- Find

a/ i^2

b/ i^3

c/ $(4-2i)+(-6+5i)$

d/ $\dfrac{-5+5i}{4-3i}$

2- Prove that $|Z_1-Z_2|^2+|Z_1+Z_2|^2=2|Z_1|^2+2|Z_2|^2$

3- Write the following in polar forms

a/ $4-3i$

b/ $\sqrt{-i}$

c/ Z^4

d/ $Z/\bar{Z}$

4- Find the value of

a/ $(1+i)^8$

b/ $(i)^{1/4}$

c/ $\sqrt{Z}$

5- Find X and y from the

$(X^2y-2)+i(X+2Xy-5)=0$

6- Find the fourth roots of (-1).

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