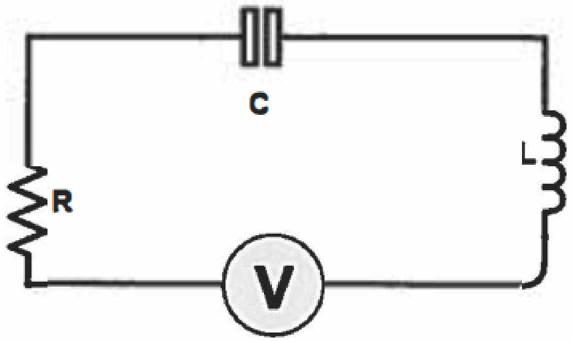
EXAMPLES ABOUT LAPLACE TRANSFORM (APPLICATIONS)

CIRCUIT ANALYSIS:

The Laplace transform actually gained its popularity from its use in analyzing electrical circuits due to Oliver Heaviside, an electrical engineer. By using Laplace transforms we can analyze an electrical circuit to discover its current, its maximum capacity and figure out if anything is wrong with the circuit. This is crucial for engineers, electrical engineers in particular, in doing their jobs to ensure the necessary machines and technology is working properly.

To start, let's show how this works in a simple RLC circuit. However, this does not mean it isn't used for more advanced types of circuits as well. For a visual aid, here is a diagram of a RLC circuit:



First let's identify the individual symbols on the circuit and what they mean. Also, while doing this it would help to identify what is used to measure each of these different pieces of the circuit for future reference. The symbols are as follows: R means resistor which is measured in ohms, L means the inductor which has inductance measured in henrys, C is the capacitor which has capacitance measured in farads and finally, V stands for the generator or battery and is measured in volts. Something to note is that another symbol commonly used for V is E when making diagrams of circuits. We can measure the charges of the capacitors and the currents by modeling them as functions of time. The equation that is used to model circuits and then subsequently used to analyze the circuits after solving it is as follows,

$$V(t) = RI + L' + \frac{1}{C}Q$$

The remaining variable left to be defined is Q, which is normally the variable used to represent the charge of a circuit. [1] We get this equation due to the fact that the voltage drop across a circuit is modeled by the following equations:

- The voltage drop across a resistor of a circuit is modeled by RI where $I = \frac{dQ}{dt}$
- Across an inductor it is modeled by $L\frac{dI}{dt}$, and since we know $I = \frac{dQ}{dt}$, we simplify this to get $L\frac{d^2Q}{dt^2}$ which we can then reduce even further to LI'.
- Across a capacitor it is modeled by $\frac{1}{C}Q$
- Across a generator it's modeled by $-V$

By taking the Laplace transform of this equation, after plugging in values for the individual pieces of the circuit, and manipulating the resulting equation to take the inverse transform we can get a final solution to our circuit.
Before we go further, it is necessary to note that when we acquired the equation for $V(t)$, we actually used Kirchhoff's Laws. [1] Due to the necessity of knowing these laws when doing circuit analysis, they are as follows:

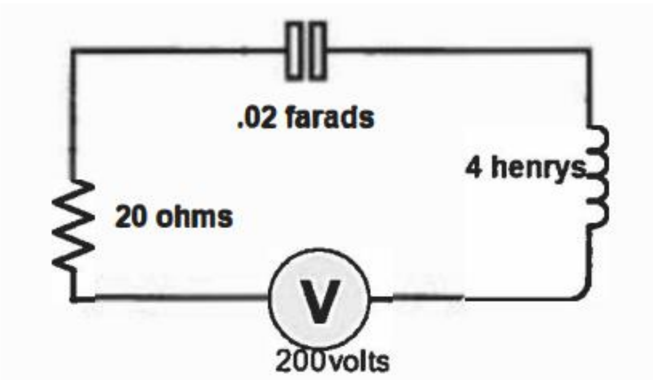
1. The algebraic sum of the currents flowing toward any junction point is equal to zero.

2. The algebraic sum of the potential drops, or the voltage drops, around any closed loop is equal to zero.

The first of these two laws is often referred to as Kirchhoff's Current Law and the second of the two as Kirchhoff's Voltage Law. These two laws are extremely important to circuit analysis, as without them, the equation that we are using to model the circuit would not work. In some cases, only of the laws needs to be applied to get the equations. However, this is usually due to it being a rather simple circuit, such as the circuit in the first example.

Now that the circuit's components have been labeled we can showcase how exactly a Laplace transform is used in an introductory example followed by a more complex example.

EXAMPLE 1:



Based on the diagram above, our circuit has an inductor of 4 henrys, a resistor of 20 ohms and a capacitor of .02 farads. As for the charge and current, let's set a condition so that the charge on the capacitor, and current in the circuit, be 0 at t=0. Let's find the charge on the capacitor at any time t besides 0, where V is equal to 200 volts. So then we get the following,

$$4\frac{dI}{dt} + 20I + \frac{1}{.02}Q = 200$$

Since $I = \frac{dQ}{dt}$,

$$4\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 50Q = 200$$

It is important to take into account that we have the following initial conditions due to our charge at $t = 0$ being 0.

1. $Q(0) = 0$
2. $Q'(0) = 0$

Now, we know the following is true

- $\frac{d^2Q}{dt^2} = Q"$
- $\frac{dQ}{dt} = Q'$

With this, we can rewrite the original equation

$$Q" + 5Q' + \frac{25}{2}Q = 50$$

Now, we take the Laplace transform

$$L\{Q" + 5Q' + \frac{25}{2}Q\} = L\{50\}$$

$$= \{s^2q - sQ(0) - Q'(0)\} + 5\{sq - Q(0)\} + \frac{25}{2}q = \frac{50}{s}$$

Recall our initial conditions to simplify this further

$$q(s^2 + 5s + 12.5) = \frac{50}{s}$$

$$= q = \frac{50}{s(s^2 + 5s + \frac{25}{2})}$$

The goal is to take the inverse Laplace transform so that we can get the answer back in the original domain of time, but as of right now it isn't clear what function we get when taking the inverse transform. Since it isn't clear what the inverse transform function would be, we need to manipulate the equation. To start is partial fraction expansion of the equation, by doing this we get

$$\frac{50}{s(s^2 + 5s + \frac{25}{2})} = \frac{A}{s} + \frac{Bs + C}{s^2 + 5s + \frac{25}{2}}$$

So, by way of doing partial fraction expansion

$$50 = A(s^2 + 5s + \frac{25}{2}) + Bs^2 + Cs$$

From here we solve for the individual variables. By plugging in 0 for s, we solve for A. Then if we plug that solution back in we can find B and C. By doing this, we end up with the following for the individual variables:

- $A = 4$
- $B = -4$
- $C = -20$

s

Now, we just plug these back into the original equation

$$= \frac{4}{s} + \frac{-4s - 20}{s^2 + 5s + \frac{25}{2}}$$

From here we manipulate the equation to fit one in the form from the table.

$$= \frac{4}{s} - 4\frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + \frac{25}{4}} - 10\frac{1}{(s + \frac{5}{2})^2 + \frac{25}{4}}$$

With the equation now fitting the table on Laplace transforms, we can take the inverse transform

$$L^{-1}\{\frac{4}{s} - 4\frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + \frac{25}{4}} - 10\frac{1}{(s + \frac{5}{2})^2 + \frac{25}{4}}\}$$

$$= 4 - 4e^{-\frac{5}{2}t}cos(\frac{5}{2}t) - 4e^{-\frac{5}{2}t}sin(\frac{5}{2}t)$$

Clarifications:

$$\frac{-4s}{s^2 + 5s + \frac{25}{2}} - \frac{20}{s^2 + 5s + \frac{25}{2}}$$

-4((s+5/2-5/2) /(s^2 +5s +25/2 +25/4 – 25/4)) - 20/(s^2 + 5s +25/2)

-4(s+5/2)/(s^2 + 5s + 25/4 +25/4)= -4(s+5/2)/(s+$(5/2)^2$)+25/4= -4cos 5/2 t $e^{-5/2\,t}$

+ 10/ (s^2 +5s + 25/2+25/4-25/4) – 20/(s^2 +5s + 25/2+25/4-25/4)

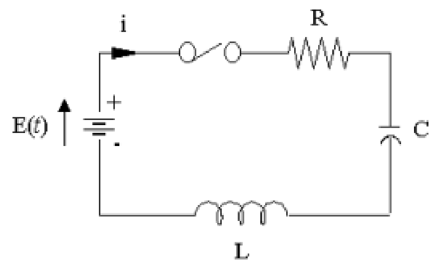
-4 cos 5/2 t $e^{-5/2\,t}$ -10/($(s + 5/2)^2$ + 25/4) multiply and divided the second term by 5/2 -4 cos 5/2 t $e^{-5/2\,t}$ – 10*2/5 * (5/2)/($(s +5/2)^2$ + 25/4)= -4 cos 5/2 t $e^{-5/2\,t}$ -4sin5/2 t. $e^{-5/2\,t}$

Solve the following differential equations:

Laplace Transform in Simple Electric Circuits:

Consider an electric circuit consisting of a resistance R, inductance L, a condenser of capacity C and electromotive power of voltage E in a series. A switch is also connected in the circuit. Then by Kirchhoff's law, we get:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E.$$



Example: An inductance of 3 henry, a resistor of 16 ohms and a capacitor of 0.02 farad are connected in series with an emf of 300 volts. At $t = 0$, the charge on the capacitor and current in the circuit is zero. Find the charge and current at any time $t > 0$.

Solution:

Let Q and I be instantaneous charge and current respectively at time t,

Then by Kirchhoff's law

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E$$

$$2\frac{d^2Q}{dt^2} + 16\frac{dQ}{dt} + 50Q = E \ldots\{\because I = \frac{dQ}{dt}$$

$$\therefore \frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q = 150$$

Applying Laplace Transform on both sides,

$$L\left[\frac{d^2Q}{dt^2}\right] + 8L\left[\frac{dQ}{dt}\right] + 25L[Q] = L[150]$$

$$\therefore \{s^2L[Q] - sQ(0) - Q'(0)\} + 8\{sL[Q] - Q(0)\} + 25L[Q] = 150L[1]$$

$$\therefore s^2L[Q] + 8\,sL[Q] + 25L[Q] = \frac{150}{s}$$

$$\therefore (s^2 + 8s + 25)L[Q] = \frac{150}{s}$$

$$\therefore \quad L[Q] = \frac{150}{s(s^2+8s+25)}$$

Taking Inverse Laplace Transform on both sides,

$$\therefore Q = L^{-1}\left[\frac{150}{s(s^2+8s+25)}\right]$$

$\therefore$ By method of partial fraction

$$Q = L^{-1}\left[\frac{6}{s} - \frac{6s+48}{(s^2+8s+25)}\right]$$

$$\therefore Q = 6L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{6(s+4)}{(s+4)^2+9}\right] - L^{-1}\left[\frac{24}{(s+4)^2+9}\right]$$

Using shifting property

$$Q = 6 - 6e^{-4t}cos3t - 8e^{-4t}sin3t$$

And $I = \frac{dQ}{dt} = 50e^{-4t}sin3t$

This is required expression for charge and current at any time $t > 0$.