عنوان المحاضرة: المتجهات Vectors

معادلة المستقيم في الفضاء Equation of line is a space

Equation of a line in a Space :—

If L is a line in space that passes through a point $P_0(x_0, y_0, z_0)$ and it parallel to a vector $V = Ai + Bj + CK$.

Then $P(x, y, z)$ is any point lies on L only if

$$\boxed{\overrightarrow{P_0P} = tV} \quad — \text{①}$$

where t is some number.

So eq(1) can be write:

$$\therefore (x - x_0)i + (y - y_0)j + (z - z_0)k = t(Ai + Bj + CK)$$

$$x - x_0 = At \; ; \; y - y_0 = Bt \; ; \; z - z_0 = Ct$$

$$\therefore \boxed{x = At + x_0} \; ; \; \boxed{y = Bt + y_0} \; ; \; \boxed{z = Ct + z_0}$$

EX:— Find parametric equations for the line through the point $(-2, 0, 4)$ parallel to the vector $V = 2i + 4j - 2K$

Solution:—

$$P_0(x_0, y_0, z_0) = (-2, 0, 4)$$

$$Aj + Bj + CK = 2i + 4j - 2K$$

$$\boxed{x = 2t - 2}$$

$$\boxed{y = 4t}$$

$$\boxed{z = -2t + 4 = 4 - 2t}$$

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Ex:- Find Parametric equations for line throug the points $P(-3,2,-3)$ and $Q(1,-1,4)$?

Solution:-

$$\overrightarrow{PQ} = (1-(-3))\,i + ((-1)-2)\,j + (4-(-3))\,K$$

$$= 4i - 3j + 7K$$

with $(x_0, y_0, z_0) = (-3, 2, -3)$

$\boxed{x = 4t-3}$ $\quad$ $\boxed{y = -3t+2}$ $\quad$ $\boxed{z = 7t-3}$

and for $(x_0, y_0, z_0) = (1, -1, 4)$

$\boxed{x = 4t+1}$ $\quad$ $\boxed{y = -3t-1}$ $\quad$ $\boxed{z = 7t+4}$

The distance from a point to a line:-

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I- The first method:-

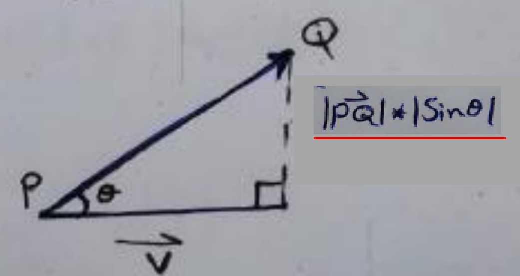
1- Find the Point $Q(x, y, z)$.

2- Calculate the distance from P to Q.

II- The second method:-

The distance from a point Q to a line that passes through a point P parallel to a vector

$$d = \frac{|\overrightarrow{PQ} \times \overrightarrow{V}|}{|\overrightarrow{V}|}$$



III- The third method:-

$$d = |\overrightarrow{PQ}| * |\sin\theta|$$

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Ex:- Find the distance from point $P(1,1,5)$ to the line $x=1+t$; $y=3-t$; $z=2t$

Solution:-

I- First method

$$Q(x,y,z)=(1+t,3-t,2t)$$

$$\vec{PQ}=(1+t-1)i+(3-t-1)j+(2t-5)k=t\,i+(2-t)j+(2t-5)$$

$$|\vec{PQ}|=\sqrt{(t)^2+(2-t)^2+(2t-5)^2}$$

$$=\sqrt{t^2+4-4t+t^2+4t^2-20t+25}$$

$$=\sqrt{6t^2-24t+29} \quad \text{بالتربيع}$$

let $(|\vec{PQ}|)^2=f(t)=d^2$

$$f(t)=6t^2-24t+29$$

$$\frac{df}{dt}=0=12t-24 \Rightarrow \boxed{t=2}$$

نعوض قيمة $(t=2)$ في معادلة $(|\vec{PQ}|)$

$$|\vec{PQ}|=\sqrt{(6*(2^2))-(24*2)+29}=\sqrt{5}=d$$

نعوض قيمة $(t=2)$ في النقطة (Q)

$$Q=(1+2,3-2,2*2)$$

$$\boxed{Q=(3,1,4)}$$

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II Second method:

$P(1,1,5)$

line: $x=1+t$, $y=3-t$, $z=2t$

$\vec{v} = i - j + 2k$

$Q=(1,3,0)$

$\vec{PQ} = (1-1)i + (3-1)j + (0-5)k$

$= 2j - 5k$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 2 & -5 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= (2*2 - (-1*-5))i - (0*2 - 1*-5)j + (0*-1 - 1*2)k$$

$$= -i - 5j - 2k$$

$$d = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(-1)^2+(-5)^2+(-2)^2}}{\sqrt{(1)^2+(-1)^2+(2)^2}}$$

$$= \frac{\sqrt{30}}{\sqrt{6}}$$

$$= \boxed{\sqrt{5}}$$

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III - Third method :-

$P=(1,1,5)$

line: $x=1+t$, $y=3-t$, $z=2t$

From eqs of line $(x=1+t ; y=3-t ; z=2t)$

$\vec{v}=i-j+2k$ & $Q=(1,3,0)$

$\overrightarrow{PQ}=0i+2j-5k$

$$\cos\theta=\frac{\overrightarrow{PQ}\cdot\vec{v}}{|\overrightarrow{PQ}|\cdot|\vec{v}|}$$

$$=\frac{(0*1)+(2*-1)+(-5*2)}{\sqrt{(0)^2+(2)^2+(-5)^2}*\sqrt{(1)^2+(-1)^2+(2)^2}}=\frac{-12}{\sqrt{29}*\sqrt{6}}$$

$\cos\theta=-0.9097 \Rightarrow \theta=\cos^{-1}(-0.9097)$

$\boxed{\theta=155.5^\circ}$

$d=|\overrightarrow{PQ}|*\sin\theta$

$=\sqrt{29}*\sin(155.5^\circ)$

$=2.2332=\boxed{\sqrt{5}}$

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Equation of the plane :-

To find the equation of the plane that passes through the Point $P_0(x_0, y_0, z_0)$ and its normal vector is $\vec{N} = ai + bj + ck$

Let $P(x, y, z)$ be any point in the plane

$$\overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$$

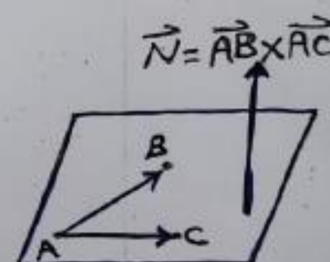
$$\overrightarrow{P_0P} \perp \vec{N} \Rightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0$$

$$\rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$\boxed{ax + by + cz = d} \Leftrightarrow \text{Equation of the Plane}$$

Ex:- Find the equation of the Plane having the points $A(2,3,5)$; $B(7,2,1)$ and $C(1,1,1)$



Solution:-

$$\overrightarrow{AB} = (7-2)i + (2-3)j + (1-5)k = \boxed{5i - j - 4k}$$

$$\overrightarrow{AC} = (1-2)i + (1-3)j + (1-5)k = \boxed{-i - 2j - 4k}$$

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 5 & -1 & -4 \\ -1 & -2 & -4 \end{vmatrix} = \boxed{-4i + 24j - 11k}$$

$\therefore -4x + 24y - 11z = d$ نعوض أحد النقاط (C,B,A) في المعادلة لايجاد قيمة (d)

$$C(1,1,1) \Rightarrow (-4*1) + (24*1) - (11*1) = d \Rightarrow \boxed{d = -9}$$

$\therefore \boxed{-4x + 24y - 11z = -9}$ is the eq of the plane.

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The distance from a point to a plane:-

If P is a point with normal $\vec{N}$, then the distance from any point (Q) to the plane is the length of the vector projection of $\vec{PQ}$ onto $\vec{N}$

$$d = \left|\text{Proj}_{\vec{N}} \vec{PQ}\right| = \left|\frac{\vec{PQ}\cdot\vec{N}}{|\vec{N}|}\right|$$

Ex:- Find the distance from the point $P(3,1,3)$ to the plane whose equation is $3X-5y+Z=4$.

Solution:-

Find any Point in the plane assume $A(0,0,4)$

$\vec{AP} = (3-0)i + (1-0)j + (3-4)K = 3i + j - K$

$\vec{N} = 3i - 5j + K$

$$d = \left|\text{Proj}_{\vec{N}} \vec{AP}\right| = \left|\frac{\vec{AP}\cdot\vec{N}}{|\vec{N}|}\right|$$

$$= \frac{|(3*3)+(1*-5)+(-1*1)|}{\sqrt{(3)^2+(-5)^2+(1)^2}}$$

$$= \boxed{\frac{3}{\sqrt{35}}}$$

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Distance between two lines :-

Note: let A be a point on line L_1
B be a point on line L_2

1- If $\vec{L_1} \times \vec{L_2} = \vec{0}$, then the two lines are parallel and the distance is defined by

$$\boxed{d = |\vec{AB}| \times |\sin\theta|} \quad \text{where } \theta = \cos^{-1}\left(\frac{\vec{AB} \cdot L_1}{|\vec{AB}| * |\vec{L_1}|}\right)$$

2- If $\vec{L_1} \times \vec{L_2} \neq 0$ and $\vec{AB} \cdot (\vec{L_1} \times \vec{L_2}) \neq 0$, then the two lines are skew-lines and the distance is defined by

$$\boxed{d = |\text{Proj}_{\vec{N}} \vec{AB}|} \quad \text{where } \vec{N} = \vec{L_1} \times \vec{L_2}$$

3- If $\vec{L_1} \times \vec{L_2} \neq 0$ and $\vec{AB} \cdot (\vec{L_1} \times \vec{L_2}) = 0$ then the two lines are intersecting and the distance between them is Zero.

$$\boxed{d = 0}$$

EX:- Find the distance between the two lines

1- $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-8}{1}$; $\frac{x}{3} = \frac{y-2}{5} = \frac{z-8}{-8}$

Solution:-

$\vec{L_1} = i - 2j + k$; $A = (1,1,8)$
$\vec{L_2} = 3i + 5j - 8k$; $B = (0,2,8)$
$\Rightarrow \vec{AB} = -i + j$

$$\vec{AB} \cdot (\vec{L_1} \times \vec{L_2}) = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 3 & 5 & -8 \end{vmatrix} = -1(-2*-8 - 5*1) - 1(1*-8 - 3*1) + 0(1*5 - 3*-2)$$

$$= -11 + 11 + 0 = 0$$

$\therefore \boxed{d = 0}$ the two lines intersect

EX:- Find the distance between the two lines

(2) $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z+3}{3}$; $\dfrac{x-2}{10} = \dfrac{y+2}{5} = \dfrac{z-3}{15}$

Solution:-

$\vec{L_1} = 2i + j + 3k$; $A = (1, 2, -3)$

$\vec{L_2} = 10i + 5j + 15k$; $B = (2, -2, 3)$

$\Rightarrow \vec{AB} = i - 4j + 6k$

$$\vec{L_1} \times \vec{L_2} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 10 & 5 & 15 \end{vmatrix} = (1*15 - 5*3)i - (2*15 - 10*3)j + (2*5 - 1*10)k$$

$$= 0i - 0j + 0k =$$

$\therefore \vec{L_1} \parallel \vec{L_2}$

$$\cos\theta = \frac{\vec{AB}\cdot\vec{L_1}}{|\vec{AB}||\vec{L_1}|} = \frac{(1*2)+(-4*1)+(6*3)}{\sqrt{(1)^2+(-4)^2+(6)^2}\sqrt{(2)^2+(1)^2+(3)^2}} = \frac{16}{\sqrt{53}\sqrt{14}}$$

$\boxed{\theta = 54.03^\circ}$

$d = |\vec{AB}| * |\sin\theta| = \sqrt{53}\,|\sin 54.03| \simeq \boxed{5.89}$ units

(3) $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z+3}{3}$; $\dfrac{x-2}{1} = \dfrac{y+2}{-1} = \dfrac{z-3}{7}$

Solution:-

$\vec{L_1} = 2i + 2j + 3k$, $A = (1, 1, -3)$

$\vec{L_2} = i - j + 7k$, $B = (2, -2, 3)$

$\Rightarrow \vec{AB} = i - 3j + 6k$

$$\vec{N} = \vec{L_1} \times \vec{L_2} = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 1 & -1 & 7 \end{vmatrix} = 17i - 11j - 4k$$

$$d = |\text{Proj}_{\vec{N}}\vec{AB}| = \frac{|\vec{AB}\cdot\vec{N}|}{|\vec{N}|} = \frac{(1*17)+(-3*-11)+(6*-4)}{\sqrt{(17)^2+(-11)^2+(-4)^2}}$$

$$= \frac{26}{\sqrt{426}} = \boxed{1.25} \text{ units}$$

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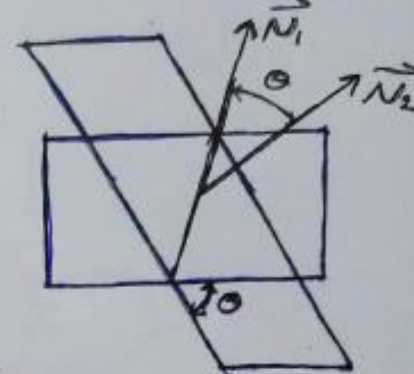
Angles between planes :-

The angle between two intersecting planes is defined to be the angle determined by their normal vectors

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}\right)}$$



Ex:- Find the angle between the planes

$3x - 6y - 2z = 15$; $2x + y - 2z = 5$

Solution:-

$\vec{N}_1 = 3i - 6j - 2k$; $\vec{N}_2 = 2i + j - 2k$

$$\theta = \cos^{-1}\left(\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}\right)$$

$$= \cos^{-1}\left(\frac{(3*2) + (-6*1) + (-2*-2)}{\sqrt{(3)^2+(-6)^2+(-2)^2} \cdot \sqrt{(2)^2+(1)^2+(-2)^2}}\right)$$

$$\therefore \theta = \cos^{-1}\left(\frac{4}{21}\right) = \boxed{79^\circ}$$

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Vector functions :-

$\vec{F}(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$

$* \; R(t) = x(t)i + y(t)j + z(t)k$

$R(t)$: is the position vector

$* \; V(t) = \frac{dR}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$

$V(t)$: is the velocity vector

$* \; a(t) = \frac{d^2R}{dt^2} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$

$a(t)$: is the acceleration vector

$* \; |V| = \sqrt{\left(\frac{dR}{dt}\right)^2} = \text{Speed}$

$\frac{V}{|V|} = \frac{dR/dt}{\sqrt{(dR/dt)^2}} = \text{Direction}$

Ex:- The vector $R(t) = (3\cos t)i + (3\sin t)j + t^2 k$ gives the position of amouving body at time. Find the body's speed and direction when $t=2$.

Solution:-

$V(t) = \frac{dR}{dt} = -(3\sin t)i + (3\cos t)j + 2t\,k$

Speed $= |V(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$

at $t=2 \Rightarrow |V(2)| = \sqrt{(-3\sin 2)^2 + (3\cos 2)^2 + (2\times 2)^2} = \boxed{5}$

Dirction $= \frac{V(2)}{|V(2)|} = \frac{1}{5}(-3\sin 2\, i + 3\cos 2\, j + 4k)$.

H.W No (2)

1- Find the acute angles between the line

(a) $3x + y = 5$; $2x - y = 4$

(b) $12x + 5y = 1$; $2x - 2y = 3$

2-

a. Find the area of the triangle determined by $P(1,1,1)$ $Q(2,1,3)$ and $R(3,-1,1)$

b. Find a unit vector perpendicular to plane PQR

3- Let $u = 5i - j + k$; $v = j - 5k$ and $w = -15i + 3j - 3k$ which vectors are (a) perpendicular (b) parallel ?

4- Find the point where the line
$x = \frac{8}{3} + 2t$, $y = -2t$ and $z = 1 + t$
intersects the plane $3x + 2y + 6z = 6$

5- Find the distance from $S(1,1,3)$ to the plane
$3x + 2y + 6z = 6$

6- Find the parametric equations for the lines for

a/ The line through the point $P(3,-4,-1)$ parallel to the vector $i + j + k$

b/ The line through $P(-2,0,3)$ and $Q(3,5,-2)$

c/ The line through the origin parallel to vector $2j + k$

d/ The line through the point $(3,-2,1)$ parallel to the line $x = 1 + 2t$, $y = 2 - t$, $z = 3t$

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