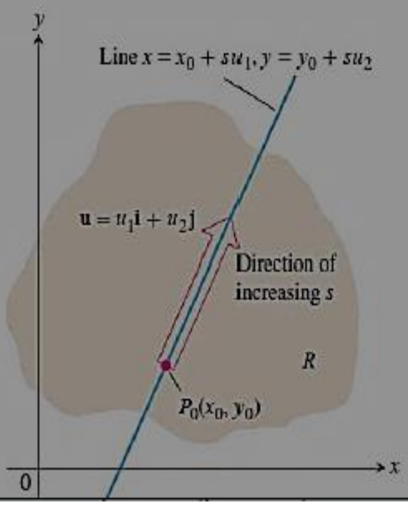
3.12 Directional derivatives

Suppose that the function $f(x,y)$ is defined throughout a region R in the xy – plane, that $P_o(x_o,y_o)$ is a point in R and that $u=u_1 i+u_2 j$ is a unit vector. Thus the equations:

$$x=x_o+su_1 \quad , \quad y=y_o+su_2$$

Parameterize the line through P_o parallel to u. if the parameter s measures arc length from P_o in the direction of u , we find the rate of change of f P_o in the direction of u by calculating $\frac{df}{ds}$ at P_o at



Gradient vector

$f(x,y,z)$ $\quad$ $P_o(x_o,y_o,z_o)$

$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

The notation ∇f is read (grad f) as well as (gradient f) and (del f).
The symbol ∇ by itself is read (del)

The directional derivatives

If $f(x,y,z)$ has continuous partial derivatives at $P_o(x_o,y_o,z_o)$ and u is a unit vector, then **the derivative of f at P_o in the direction of u** is:

$$(\boldsymbol{DuF})|p_\circ = \boldsymbol{\nabla f}|p_\circ . \boldsymbol{u}$$

Which is the scalar product of the gradient of F at $P_\circ$ and u

"$(\boldsymbol{DuF})|p_\circ \rightarrow$ The derivative of F at P_o in the direction of u"

Example: Find the direction derivative of the function

F(x,y,z)=$x^2 + y^2 + z^2$ *at point* $p_\circ(1,1,1)$ *in the direction of*

Vector v=i+j+k .

Solution //

$$(\boldsymbol{DuF})|p_\circ = \boldsymbol{\nabla f}|p_\circ . \boldsymbol{u}$$

$$\boldsymbol{\nabla f} = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\boldsymbol{k} = \mathbf{2}\boldsymbol{x}\,\boldsymbol{i} + \mathbf{2}\boldsymbol{y}\,\boldsymbol{j} + \mathbf{2}\boldsymbol{z}\,\boldsymbol{k}$$

$$\boldsymbol{at\ point\ p_\circ(1,1,1)} \quad \rightarrow \quad \boldsymbol{\nabla f}|p_\circ = 2i + 2j + 2k$$

$$\mathbf{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{\boldsymbol{i+j+k}}{\sqrt{\mathbf{1+1+1}}} = \frac{\boldsymbol{i+j+k}}{\sqrt{\mathbf{3}}}$$

$$(\boldsymbol{DuF})|\boldsymbol{p_\circ} = \boldsymbol{\nabla f}|\boldsymbol{p_\circ}\,.\,\boldsymbol{u} = \mathbf{2}\boldsymbol{i} + \mathbf{2}\boldsymbol{j} + \mathbf{2}\boldsymbol{k}\ .\ \frac{\boldsymbol{i+j+k}}{\sqrt{\mathbf{3}}} = \frac{\mathbf{6}}{\sqrt{\mathbf{3}}}$$

Example: find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2\ ,0)$ in the direction of $v = 3i - 4j$

Solution: the direction of v is the unit vector obtained by dividing v by its length:

$$u = \frac{v}{|v|}$$

$$|v| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$u = \frac{3i-4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$\boldsymbol{\nabla f} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{i} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{j} = (\boldsymbol{e^y} - \boldsymbol{y}\ \boldsymbol{\sin xy})\,\boldsymbol{i} + (\ \boldsymbol{xe^y} - \boldsymbol{x}\,\boldsymbol{\sin xy})\,\boldsymbol{j}$$

$$\boldsymbol{at\ point\ p_\circ(2,0)} \quad \rightarrow \quad \boldsymbol{\nabla f}|p_\circ = i + 2j$$

$$(\boldsymbol{DuF})|\boldsymbol{p_\circ} = \boldsymbol{\nabla f}|\boldsymbol{p_\circ}\,.\,\boldsymbol{u} = \boldsymbol{i} + \mathbf{2}\boldsymbol{j}.\ \frac{\mathbf{3}\boldsymbol{i}-\mathbf{4}\boldsymbol{j}}{\mathbf{5}} = \frac{-\mathbf{5}}{\mathbf{5}} = -\mathbf{1}$$

Example: find the derivative of function $f(x,y) = 2xy - 3y^2$ at the point $P_o\,(5\,,5)$ in the direction of $A = 4i + 3j$

Solution:

$$u = \frac{A}{|A|}$$

$$|A| = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$u = \frac{4i+3j}{5} = \frac{4}{5}i + \frac{3}{5}j$$

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial z}j = (2y)i + (\,2x - 6y)j$$

$$at\ point\ p_\circ(5,5) \quad \rightarrow \quad \nabla f|p_\circ = 10i - 20j$$

$$(DuF)|p_\circ = \nabla f|p_\circ\,.\,u = 10i - 20j.\ \frac{4i+3j}{5} = \frac{-20}{5} = -4$$

Exercises:

1. find the derivative of the function $f(x,y,z) = xy + yz + zx$,at the point $P_0(1,-1,2)$ in the direction of $A = 3i + 6j - 2k$

2. find the derivative of the function $g(x,y,z) = 3e^x \cos yz$,at the point $P_0(0,0,0)$ in the direction of $A = 2i + j - 2k$