Chapter three
Derivatives

Let $y = f(x)$ be a function of x . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 – **Find the derivative of the function :** $f(x) = \frac{1}{\sqrt{2x+3}}$

Sol.:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

<u>Rules of derivatives</u> : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$
3. $\frac{d}{dx} cu = c\frac{du}{dx}$
4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$

and $\frac{d}{dx}(u.v.w) = u.v\frac{dw}{dx} + u.w\frac{dv}{dx} + v.w\frac{du}{dx}$

6. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ *where* $v \neq 0$

EX-2- **Find** $\frac{dy}{dx}$ **for the following functions :**

$a)\ y = (x^2+1)^5$

$b)\ y = [(5-x)(4-2x)]^2$

$c)\ y = (2x^3 - 3x^2 + 6x)^{-5}$

$d)\ y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

$e)\ y = \frac{(x^2+x)(x^2-x+1)}{x^3}$

$f)\ y = \frac{x^2-1}{x^2+x-2}$

Sol.-

$a)\ \frac{dy}{dx} = 5(x^2+1)^4 .2x = 10x(x^2+1)^4$

$b)\ \frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$= 8(5-x)(2-x)(2x-7)$

$c)\ \frac{dy}{dx} = -5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$= -30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\ y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$

$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$

$e)\ y = \frac{(x+1)(x^2-x+1)}{x^3} \Rightarrow$

$\frac{dy}{dx} = \frac{x^3[(x^2-x+1)+(x+1)(2x-1)] - 3x^2(x+1)(x^2-x+1)}{x^6} = -\frac{3}{x^4}$

$f)\ \frac{dy}{dx} = \frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2} = \frac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule:

1. **Suppose that $h = g_o f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x, then y is a differentiable function of x :**

$$y = g(t) \;\; and \;\; t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 **– Use the chain rule to express dy/dx in terms of x and y :**

$a)\quad y = \dfrac{t^2}{t^2+1} \qquad and \quad t = \sqrt{2x+1}$

$b)\quad y = \dfrac{1}{t^2+1} \qquad and \quad x = \sqrt{4t+1}$

$c)\quad y = \left(\dfrac{t-1}{t+1}\right)^2 \qquad and \quad x = \dfrac{1}{t^2} - 1 \quad at \quad t = 2$

$d)\quad y = 1 - \dfrac{1}{t} \qquad and \quad t = \dfrac{1}{1-x} \quad at \quad x = 2$

Sol.-

$$a)\quad y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where}\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where}\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow \frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\qquad \text{at}\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

Higher derivatives **: If a function $y = f(x)$ possesses a derivative at every point of some interval , we may form the function $f'(x)$ and talk**

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y'' ,\ f''(x) ,\ or\ \frac{d^2f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)}, f^{(n)}(x), \frac{d^ny}{dx^n} .$$

EX-4- **Find all derivatives of the following function :**

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol.-

$$\frac{dy}{dx}=9x^2-8x+7 \quad , \quad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \quad , \quad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

Ex-5 – **Find the third derivative of the following function :**

$$y=\frac{1}{x}+\sqrt{x^3}$$

Sol.-

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation: **If the formula for f is an algebraic combination of powers of x and y . To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to x .**

EX-6- **Find $\dfrac{dy}{dx}$ for the following functions:**

$a)\ x^2 . y^2 = x^2 + y^2$

$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \dfrac{x-y}{x-2y} = 2 \ \ at\ \ P(3,1)$

$d)\ xy + 2x - 5y = 2 \ \ at\ P(3,2)$

Sol.

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

Exponential functions : **If u is any differentiable function of x , then :**

$$7)\quad \frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u . \frac{du}{dx}$$

EX-7 – Find $\dfrac{dy}{dx}$ for the following functions :

$a)\ y = 2^{3x}$ $\qquad b)\ y = 2^{x}.3^{x}$

$c)\ y = (2^{x})^{2}$ $\qquad d)\ y = x.2^{x^2}$

$e)\ y = e^{(x+e^{5x})}$ $\qquad f)\ y = e^{\sqrt{1+5x^2}}$

Sol.-

$$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3\ln 2$$

$$b)\ y = 2^{x}.3^{x} \Rightarrow y = 6^{x} \Rightarrow \frac{dy}{dx} = 6^{x}.\ln 6$$

$$c)\ y = (2^{x})^{2} \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}\ln 2.2 = 2^{2x+1}\ln 2$$

$$d)\ y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2}\ln 2.2x + 2^{x^2} = 2^{x^2}(2x^2\ln 2 + 1)$$

$$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1+5e^{5x})$$

$$f)\ y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\frac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\ \frac{5x}{\sqrt{1+5x^2}}$$

Logarithm functions : If u is any differentiable function of x , then :

$$8)\ \frac{d}{dx}\log_a u = \frac{1}{u.\ln a}.\frac{du}{dx} \qquad \text{and} \qquad \frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

EX-8 – Find $\dfrac{dy}{dx}$ for the following functions :

$a)\ y = \log_{10} e^{x}$ $\qquad b)\ y = \log_5 (x+1)^2$

$c)\ y = \log_2 (3x^2+1)^3$ $\qquad d)\ y = \left[\ln(x^2+2)^2\right]^3$

$e)\ y + \ln(xy) = 1$ $\qquad f)\ y = \dfrac{(2x^3-4)^{\frac{2}{3}}.(2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

Sol. –

$$a)\ y=\log_{10}e^{x} \Rightarrow y=x\log_{10}e \Rightarrow \frac{dy}{dx}=\log_{10}e=\frac{\ln e}{\ln 10}=\frac{1}{\ln 10}$$

$$b)\ y=\log_5(x+1)^2=2\log_5(x+1) \Rightarrow \frac{dy}{dx}=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=3\log_2(3x^2+1) \Rightarrow \frac{dy}{dx}=\frac{3}{3x^2+1}\cdot\frac{6x}{\ln 2}=\frac{18x}{(3x^2+1)\ln 2}$$

$$d)\ \frac{dy}{dx}=3\left[2\ln(x^2+2)\right]^2\frac{2}{x^2+2}.2x=\frac{48x\left[\ln(x^2+2)\right]^2}{x^2+2}$$

$$e)\ y+\ln x+\ln y=1 \Rightarrow \frac{dy}{dx}+\frac{1}{x}+\frac{1}{y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x(y+1)}$$

$$f)\ \ln y=\frac{2}{3}\ln(2x^3-4)+\frac{5}{2}\ln(2x^2+3)-2\ln(7x^3+4x-3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{3}.\frac{6x^2}{2x^3-4}+\frac{5}{2}.\frac{4x}{2x^2+3}-2.\frac{21x^2+4}{7x^3+4x-3}$$

$$\Rightarrow \frac{dy}{dx}=2y\left[\frac{2x^2}{2x^3-4}+\frac{5x}{2x^2+3}-\frac{21x^2+4}{7x^3+4x-3}\right]$$

<u>*Trigonometric functions*</u> *:* If *u* is any differentiable function of *x* , then :

$$9)\ \frac{d}{dx}\sin u=\cos u.\frac{du}{dx}$$

$$10)\ \frac{d}{dx}\cos u=-\sin u.\frac{du}{dx}$$

$$11)\ \frac{d}{dx}\tan u=\sec^2 u.\frac{du}{dx}$$

$$12)\ \frac{d}{dx}\cot u=-\csc^2 u.\frac{du}{dx}$$

$$13)\ \frac{d}{dx}\sec u=\sec u.\tan u.\frac{du}{dx}$$

$$14)\ \frac{d}{dx}\csc u=-\csc u.\cot u.\frac{du}{dx}$$

<u>*EX-9*</u>- Find $\frac{dy}{dx}$ for the following functions :

$a)\ y = \tan(3x^2)$ $\qquad$ $b)\ y = (\csc x + \cot x)^2$

$c)\ y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$ $\qquad$ $d)\ y = \tan^2(\cos x)$

$e)\ x + \tan(xy) = 0$ $\qquad$ $f)\ y = \sec^4 x - \tan^4 x$

Sol.-

$$a)\ \frac{dy}{dx} = \sec^2(3x^2).6x = 6x.\sec^2(3x^2)$$

$$b)\ \frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x.\cot x - \csc^2 x) = -2\csc x.(\csc x + \cot x)^2$$

$$c)\ \frac{dy}{dx} = 2\cos\frac{x}{2}.\frac{1}{2} - \left[x(-\sin\frac{x}{2}).\frac{1}{2} + \cos\frac{x}{2}\right] = \frac{x}{2}.\sin\frac{x}{2}$$

$$d)\ \frac{dy}{dx} = 2.\tan(\cos x).\sec^2(\cos x).(-\sin x) = -2.\sin x.\tan(\cos x).\sec^2(\cos x)$$

$$e)\ 1 + \sec^2(xy).(x\frac{dy}{dx} + y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y.\sec^2(xy)}{x.\sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$$

$$f)\ \frac{dy}{dx} = 4\sec^3 x.\sec x.\tan x - 4.\tan^3 x.\sec^2 x = 4\tan x.\sec^2 x$$

EX-10- **Prove that :**

$$a)\ \frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx} \qquad b)\ \frac{d}{dx}\sec u = \sec u.\tan u.\frac{du}{dx}$$

Proof :

$$a)\quad L.H.S. = \frac{d}{dx}\tan u = \frac{d}{dx}\frac{\sin u}{\cos u} = \frac{\cos u.\cos u.\frac{du}{dx} - \sin u.(-\sin u)\frac{du}{dx}}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}.\frac{du}{dx} = \frac{1}{\cos^2 u}.\frac{du}{dx} = \sec^2 u.\frac{du}{dx} = R.H.S.$$

$$b)\quad L.H.S. = \frac{d}{dx}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{1}{\cos u}.\frac{\sin u}{\cos u}.\frac{du}{dx} = \sec u.\tan u.\frac{du}{dx} = R.H.S.$$

The inverse trigonometric functions : **If *u* is any differentiable function**

of x , then :

15) $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$

16) $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$

17) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

18) $\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$

19) $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$

20) $\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$

<u>*EX-11*</u>- Find $\frac{dy}{dx}$ in each of the following functions :

a) $y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$ b) $y = \sin^{-1}\frac{x-1}{x+1}$

c) $y = x.\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$ d) $y = \sec^{-1}5x$

e) $y = x.\ln(\sec^{-1}x)$ f) $y = 3^{\sin^{-1}2x}$

<u>*Sol.*</u> –

a) $\frac{dy}{dx} = -\frac{1}{1+\left(\frac{2}{x}\right)^2}2.\left(-\frac{1}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2}.\frac{1}{2} = \frac{4}{4+x^2}$

b) $\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}.\frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$

c) $\frac{dy}{dx} = x\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{1}{4}.\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1}2x$

d) $\frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$

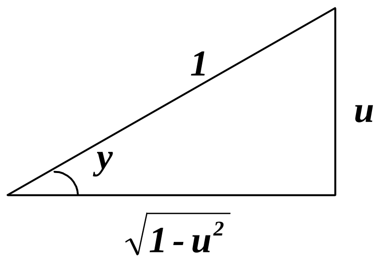
$$e)\quad \frac{dy}{dx} = \frac{x}{sec^{-1}x}\frac{1}{|x|\sqrt{x^2-1}} + ln(sec^{-1}x) = \frac{1}{\sqrt{x^2-1}\,.sec^{-1}x} + ln(sec^{-1}x)$$

$$f)\quad \frac{dy}{dx} = 3^{sin^{-1}2x}.ln3.\frac{2}{\sqrt{1-4x^2}}$$

<u>*EX-12*</u>- **Prove that :**

$$a)\ \frac{d}{dx}sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad b)\ \frac{d}{dx}tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

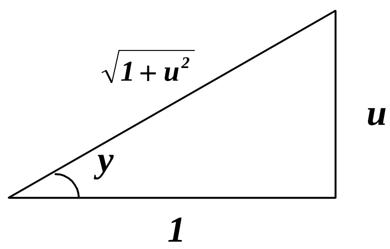
<u>Proof</u> : a)



$$Let\ \ y = sin^{-1}u \Rightarrow u = sin\,y \Rightarrow \frac{du}{dx} = cos\,y.\frac{dy}{dx} = \sqrt{1-u^2}\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \Rightarrow \frac{d}{dx}sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

b)



$$Let\ y = tan^{-1}u \Rightarrow u = tany \Rightarrow \frac{du}{dx} = sec^2y\,.\frac{du}{dx} = \left(\sqrt{1+u^2}\right)^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2}.\frac{du}{dx} \Rightarrow \frac{d}{dx}tan^{-1}u = \frac{1}{1+u^2}.\frac{du}{dx}$$

Hyperbolic functions **: If *u* is any differentiable function of *x* , then :**

$$21)\quad \frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$$

$$22)\quad \frac{d}{dx}\cosh u = \sinh u.\frac{du}{dx}$$

$$23)\quad \frac{d}{dx}\tanh u = \operatorname{sech}^2 u.\frac{du}{dx}$$

$$24)\quad \frac{d}{dx}\coth u = -\operatorname{csch}^2 u.\frac{du}{dx}$$

$$25)\quad \frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u.\tanh u.\frac{du}{dx}$$

$$26)\quad \frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u.\coth u.\frac{du}{dx}$$

EX-13 **- Find $\frac{dy}{dx}$ for the following functions :**

$a)\ y = \coth(\tan x)$ $\qquad b)\ y = \sin^{-1}(\tanh x)$

$c)\ y = \ln\left|\tanh\frac{x}{2}\right|$ $\qquad d)\ y = x.\sinh 2x - \frac{1}{2}.\cosh 2x$

$e)\ y = \operatorname{sech}^3 x$ $\qquad f)\ y = \operatorname{csch}^2 x$

Sol. **-**

$$a)\ \frac{dy}{dx} = -\operatorname{csch}^2(\tan x).\sec^2 x$$

$$b)\ \frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

$$c)\ \frac{dy}{dx} = \frac{1}{\tanh\frac{x}{2}}\operatorname{sech}^2\frac{x}{2}.\frac{1}{2} = \frac{\dfrac{1}{\cosh^2\frac{x}{2}}}{2.\dfrac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}}$$

$$= \frac{1}{2\sinh\frac{x}{2}.\cosh\frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{csch} x$$

$$d)\ \frac{dy}{dx} = x\cosh 2x.2 + \sinh 2x - \frac{1}{2}\sinh 2x.2 = 2x\cosh 2x$$
$$e)\ \frac{dy}{dx} = 3\,sech^2 x(-sech\, x.\tanh x) = -3\,sech^3 x.\tanh x$$
$$f)\ \frac{dy}{dx} = 2\,csch\, x(-csch\, x.\coth x) = -2\,csch^2 x.\coth x$$

EX-14- **Show that the functions :**

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \quad and \quad y = \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}}$$

Taken together , satisfy the differential equations :

$$i)\ \frac{dx}{dt} + 2\frac{dy}{dt} + x = 0 \quad and \quad ii)\ \frac{dx}{dt} - \frac{dy}{dt} + y = 0$$

Proof -

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \Rightarrow \frac{dx}{dt} = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}}$$
$$y = \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}} \Rightarrow \frac{dy}{dt} = \frac{1}{3}\cosh\frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}$$

$$i)\ \frac{dx}{dt} + 2\frac{dy}{dt} + x = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}} + \frac{2}{3}\cosh\frac{t}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} - \frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} = 0$$
$$ii)\ \frac{dx}{dt} - \frac{dy}{dt} + y = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}} - \frac{1}{3}\cosh\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}} = 0$$

EX-15 - **Prove that :**

$$a)\ \frac{d}{dx}\tanh u = sech^2 u.\frac{du}{dx} \quad and \quad b)\ \frac{d}{dx}sech\, u = -sech\, u.\tanh u.\frac{du}{dx}$$

Proof-

$$a)\ \frac{d}{dx}\tanh u = \frac{d}{dx}\left(\frac{\sinh u}{\cosh u}\right) = \frac{\cosh u.\cosh u.\frac{du}{dx} - \sinh u.\sinh u.\frac{du}{dx}}{\cosh^2 u}$$
$$= \frac{(\cosh^2 u - \sinh^2 u)\frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u}.\frac{du}{dx} = sech^2 u.\frac{du}{dx}$$

$$b)\ \frac{d}{dx}\frac{1}{\cosh u} = -\frac{1}{\cosh^2 u}.\sinh u.\frac{du}{dx} = -sech\, u.\tanh u.\frac{du}{dx}$$

The inverse hyperbolic functions **: If *u* is any differentiable function of *x* , then :**

27) $$\frac{d}{dx}\sinh^{-1}u=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

28) $$\frac{d}{dx}\cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

29) $$\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx} \qquad |u|<1$$

30) $$\frac{d}{dx}\coth^{-1}u=\frac{1}{1-u^2}\frac{du}{dx} \qquad |u|>1$$

31) $$\frac{d}{dx}\operatorname{sech}^{-1}u=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$$

32) $$\frac{d}{dx}\operatorname{csch}^{-1}u=-\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

EX-16 **- Find $\frac{dy}{dx}$ for the following functions :**

a) $y=\cosh^{-1}(\sec x)$ b) $y=\tanh^{-1}(\cos x)$

c) $y=\coth^{-1}(\sec x)$ d) $y=\operatorname{sech}^{-1}(\sin 2x)$

Sol.-

a) $$\frac{dy}{dx}=\frac{\sec x.\tan x}{\sqrt{\sec^2 x-1}}=\frac{\sec x.\tan x}{\sqrt{\tan^2 x}}=\sec x \qquad where \quad \tan x>0$$

b) $$\frac{dy}{dx}=\frac{-\sin x}{1-\cos^2 x}=\frac{-\sin x}{\sin^2 x}=-\csc x$$

c) $$\frac{dy}{dx}=\frac{\sec x.\tan x}{1-\sec^2 x}=\frac{\sec x.\tan x}{-\tan^2 x}=-\csc x$$

d) $$\frac{dy}{dx}=-\frac{2.\cos 2x}{\sin 2x.\sqrt{1-\sin^2 2x}}=-2\csc 2x \qquad where \quad \cos 2x>0$$

EX-17 **– Verify the following formulas :**

a) $$\frac{d}{dx}\cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}$$

b) $$\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}.\frac{du}{dx} \qquad |u|<1$$

Proof

a) Let $y = cosh^{-1} u \Rightarrow u = cosh\, y$

$$\frac{du}{dx} = sinh\, y.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{sinh\, y}.\frac{du}{dx}$$

$$cosh^2 y - sinh^2 y = 1 \Rightarrow u^2 - sinh^2 y = 1 \Rightarrow sinh\, y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}}.\frac{du}{dx} \Rightarrow \frac{d}{dx} cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}.\frac{du}{dx}$$

b) Let $y = tanh^{-1} u \Rightarrow u = tanh\, y$

$$\frac{du}{dx} = sech^2 y.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{sech^2 y}.\frac{du}{dx}$$

$$sech^2 y + tanh^2 y = 1 \Rightarrow sech^2 y + u^2 = 1 \Rightarrow sech^2 y = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{1 - u^2}.\frac{du}{dx} \Rightarrow \frac{d}{dx} tanh^{-1} u = \frac{1}{1 - u^2}.\frac{du}{dx}$$

The derivatives of functions like u^v *:* **Where *u* and *v* are differentiable functions of x , are found by logarithmic differentiation :**

Let $y = u^v \Rightarrow ln\, y = v.ln\, u$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{v}{u}.\frac{du}{dx} + ln\, u.\frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{v}{u}.\frac{du}{dx} + ln\, u.\frac{dv}{dx}\right]$$

33) $$\frac{d}{dx} u^v = u^v.\left[\frac{v}{u}.\frac{du}{dx} + ln\, u.\frac{dv}{dx}\right]$$

EX-18- **Find $\frac{dy}{dx}$ for :**

a) $y = x^{cosx}$ b) $y = (lnx + x)^{tanx}$

Sol. -

a) $y = x^{cos\, x} \Rightarrow ln\, y = cos\, x.ln\, x \Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{cos\, x}{x} + ln\, x.(-sin\, x)$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{cos\, x}{x} - sin\, x.ln\, x\right]$$

or by formula , where u = x and v = cosx

$$\frac{dy}{dx} = y\left[\frac{cos\, x}{x} - sin\, x.ln\, x\right]$$

$b)\quad y=(\ln x+x)^{\tan x}\Rightarrow \ln y=\tan x.\ln(\ln x+x)$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{\tan x}{\ln x+x}.(\frac{1}{x}+1)+\ln(\ln x+x).\sec^2 x$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{(x+1).\tan x}{x(\ln x+x)}+\ln(\ln x+x).\sec^2 x\right]$$

or by formula, where $u=\ln x+x$ *and* $v=\tan x$

$$\frac{dy}{dx}=y.\left[\frac{\tan x}{\ln x+x}(\frac{1}{x}+1)+\ln(\ln x+x).\sec^2 x\right]$$

$$=y.\left[\frac{(x+1).\tan x}{x(\ln x+x)}+\ln(\ln x+x).\sec^2 x\right]$$

Problems -3

1. Find $\frac{dy}{dx}$ for the following functions :

1) $y=(x-3)(1-x)$ $\quad (ans.: 4-2x)$

2) $y=\frac{ax+b}{x}$ $\quad (ans.: -\frac{b}{x^2})$

3) $y=\frac{3x+4}{2x+3}$ $\quad (ans.: \frac{1}{(2x+3)^2})$

4) $y=3x^3-2\sqrt{x}+\frac{5}{x^2}$ $\quad (ans.: 9x^2-\frac{1}{\sqrt{x}}-\frac{10}{x^3})$

5) $y=\left(\sqrt{x^3}-\frac{1}{\sqrt{x^3}}\right)^2$ $\quad (ans.: \frac{3(x^6-1)}{x^4})$

6) $y=(2x-1)^2(3x+2)^3+\frac{1}{(x-2)^2}$ $\quad (ans.: (2x-1)(3x+2)^2(30x-1)-\frac{2}{(x-2)^3})$

7) $y=ln(ln x)$ $\quad (ans.: \frac{1}{x.ln x})$

8) $y=ln(Cosx)$ $\quad (ans.: -tan x)$

9) $y=Sinx^3$ $\quad (ans.: 3x^2.Cosx^3)$

10) $y=Cos^{-3}(5x^2+2)$ $\quad (ans.: \frac{30x.Sin(5x^2+4)}{Cos^4(5x^2+4)})$

11) $y=tan x.sin x$ $\quad (ans.: Sinx+tan x.Secx)$

12) $y=tan(Secx)$ $\quad (ans.: Sec^2(Secx).Secx.tan x)$

13) $y=Cot^3\left(\frac{x+1}{x-1}\right)$ $\quad (ans.: \frac{6}{(x-1)^2}.Cot^2\left(\frac{x+1}{x-1}\right).Csc^2\left(\frac{x+1}{x-1}\right))$

14) $y=\frac{Cosx}{x}$ $\quad (ans.: -\frac{x.Sinx+Cosx}{x^2})$

15) $y=\sqrt{tan\sqrt{2x+7}}$ $\quad (ans.: \frac{Sec^2\sqrt{2x+7}}{2\sqrt{2x+7}\sqrt{tan\sqrt{2x+7}}})$

16) $y=x^2.Sinx$ $\quad (ans.: x^2.Cosx+2x.Sinx)$

17) $y=Csc^{-\frac{2}{3}}\sqrt{5x}$ $\quad (ans.: \frac{5}{3\sqrt{5x}}.\frac{Cot\sqrt{5x}}{Csc^{\frac{2}{3}}\sqrt{5x}})$

18) $y=x\left[Sin(ln x)+Cos(ln x)\right]$ $\quad (ans.: 2.Cos(ln x))$

19) $y = Sin^{-1}(5x^2)$ $\quad (ans.: \dfrac{10x}{\sqrt{1-25x^4}})$

20) $y = Cot^{-1}\left(\dfrac{1+x}{1-x}\right)$ $\quad (ans.: -\dfrac{1}{1+x^2})$

21) $y = tan^{-1}\sqrt{4x^3-2}$ $\quad (ans.: \dfrac{6x^2}{(4x^3-1)\sqrt{4x^3-2}})$

22) $y = Sec^{-1}(3x^2+1)^3$ $\quad (ans.: \dfrac{18x}{|3x^2+1|\sqrt{(3x^2+1)^6-1}})$

23) $y = Sin^{-1}\dfrac{x^2}{2-x} + x^2.Sec^{-1}\dfrac{x}{2}$ $\quad (ans.: \dfrac{4x-x^2}{(2-x)\sqrt{(2-x)^2-x^4}} + \dfrac{2x}{\sqrt{x^2-4}} + 2x.Sec^{-1}\dfrac{x}{2})$

24) $y = Sin^{-1}2x.Cos^{-1}2x$ $\quad (ans.: \dfrac{2(Cos^{-1}2x - Sin^{-1}2x)}{\sqrt{1-4x^2}})$

25) $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$ $\quad (ans.: \dfrac{y}{3}\left[\dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1} - \dfrac{2}{2x+3}\right])$

26) $y = tan^{-1}(\ln x)$ $\quad (ans.: \dfrac{1}{x(1+(\ln x)^2)})$

27) $y^{\frac{4}{3}} = \dfrac{\sqrt{\sin x.\cos x}}{1+2.\ln x}$ $\quad (ans.: \dfrac{3y}{4}(\dfrac{\cot x}{2} - \dfrac{\tan x}{2} - \dfrac{2}{x(1+2\ln x)}))$

28) $\sqrt{y} = \dfrac{x^5.tan^{-1}x}{(3-2x).\sqrt[3]{x}}$ $\quad (ans.: 2y(\dfrac{14}{3x} + \dfrac{1}{(1+x^2).tan^{-1}x} + \dfrac{2}{3-2x}))$

29) $y = sec^{-1}e^{2x}$ $\quad (ans.: \dfrac{2}{\sqrt{e^{4x}-1}})$

30) $y = (\cos x)^{\sqrt{x}}$ $\quad (ans.: \dfrac{y}{2\sqrt{x}}(\ln\cos x - 2x.\tan x))$

31) $y = (\sin x)^{\tan x}$ $\quad (ans.: y(1+sec^2 x.\ln\sin x))$

32) $y = \sqrt{2x^2 + cosh^2(5x)}$ $\quad (ans.: \dfrac{2x + 5cosh(5x).sinh(5x)}{\sqrt{2x^2+cosh^2(5x)}})$

33) $y = sinh(\cos 2x)$ $\quad (ans.: -2\sin 2x.cosh(\cos 2x))$

34) $y = csc\,h\dfrac{1}{x}$ $\quad (ans.: \dfrac{1}{x^2}.csc\,h\dfrac{1}{x}.coth\dfrac{1}{x})$

35) $y = x^{2.}.tanh^2\sqrt{x}$ $\quad (ans.: x.tanh\sqrt{x}(\sqrt{x}\,sec\,h^2\sqrt{x} + 2tanh\sqrt{x}))$

$36) \quad y = ln\frac{sin x.cos x + tan^3 x}{\sqrt{x}}$ $\quad (ans.: \frac{cos^2 x - sin^2 x + 3tan^2 x.sec^2 x}{sin x.cos x + tan^3 x} - \frac{1}{2x})$

$37) \quad y = log_4 sin x$ $\quad (ans.: \frac{cot x}{ln4})$

$38) \quad y = e^{(x^2 - e^{5x})}$ $\quad (ans.: (2x - 5e^{5x})e^{(x^2 - e^{5x})})$

$39) \quad y = e^{x^2 tan x}$ $\quad (ans.: (x^2 sec^2 x + 2x tan x)e^{x^2 tan x})$

$40) \quad y = 7^{csc\sqrt{2x+3}}$ $\quad (ans.: \frac{-7^{csc\sqrt{2x+3}} ln7}{\sqrt{2x+3}} csc\sqrt{2x+3}.cot\sqrt{2x+3})$

$41) \quad y = \left[ln(x^2 + 2)^2\right]cos x$ $\quad (ans.: \frac{4x.cos x}{x^2 + 2} - 2ln(x^2 + 2)sin x)$

$42) \quad y = sinh^{-1}(tan x)$ $\quad (ans.: |sec x|)$

$43) \quad y = \sqrt{1 + (ln x)^2}$ $\quad (ans.: \frac{ln x}{x\sqrt{1 + (ln x)^2}})$

$44) \quad y = \frac{e^x}{ln x}$ $\quad (ans.: \frac{e^x(x ln x - 1)}{x(ln x)^2})$

$45) \quad y = x^3 log_2(3 - 2x)$ $\quad (ans.: 3x^2 log_2(3 - 2x) - \frac{2x^3}{(3 - 2x)ln2})$

$46) \quad y = 2cosh^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{x^2 - 4}$ $\quad (ans.: \frac{x^2}{\sqrt{x^2 - 4}})$

2. Verify the following derivatives :

$a) \quad \frac{d}{dx}\left[5x + (\sqrt{x} + \frac{1}{\sqrt{x}})^2\right] = 6 - \frac{1}{x^2}$

$b) \quad \frac{d}{dx}\left[\sqrt{x}(ax^2 + bx + c)\right] = \frac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

3. Find the derivative of y with respect to x in the following functions :

$a) \quad y = \frac{u^2}{u^2 + 1} \quad and \quad u = 3x^3 - 2$ $\quad (ans.: \frac{18x^2 y^2}{(3x^3 - 2)^3})$

$b) \quad y = \sqrt{u} + 2u \quad and \quad u = x^2 - 3$ $\quad (ans.: \frac{x}{\sqrt{x^2 - 3}} + 4x)$

4. Find the second derivative for the following functions :

a) $y = (x + \frac{1}{x})^3$ $\quad (ans.: 6x + \frac{6}{x^3} + \frac{12}{x^5})$

b) $f(x) = \sqrt{2x} + \frac{2\sqrt{2}}{\sqrt{x}}$ at $x = 2$ $\quad (ans.: \frac{1}{4})$

c) $x^2 - 2xy + y^2 - 16x = 0$ $\quad (ans.: \mp x^{-\frac{3}{2}})$

5. Find the third derivative of the function :

$y = \sqrt{x^3}$ $\quad (ans.: -\frac{3}{8y})$

6. Show for $y = \frac{u}{v}$ **that** $y'' = \frac{v(vu'' - uv'') - 2v'(vu' - uv')}{v^3}$.

7. Show for $y = u.v$ **that** $y''' = uv''' + 3u'v'' + 3u''v' + u'''v$.

8. Show that $y = 35x^4 - 30x^2 + 3$ **satisfies** $(1 - x^2)y'' - 2xy' + 20y = 0$.

9. Find $\frac{dy}{dx}$ **for the following implicit functions :**

a) $x^3 + 4x\sqrt{y} - \frac{5y^2}{x} = 3$ $\qquad (ans.: \frac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - \frac{2x}{\sqrt{y}}})$

b) $\sqrt{xy} + 1 = y$ $\qquad (ans.: \frac{y}{2\sqrt{xy} - x})$

c) $3xy = (x^3 + y^3)^{\frac{3}{2}}$ $\qquad (ans.: \frac{3x^2\sqrt{x^3 + y^3} - 2y}{2x - 3y^2\sqrt{x^3 + y^3}})$

d) $x^3 + x.\tan^{-1} y = y$ $\qquad (ans.: \frac{(1+y^2)(3x^2 + \tan^{-1} y)}{1 + y^2 - x})$

e) $\sin^{-1}(xy) = \cos^{-1}(x - y)$ $\qquad (ans.: \frac{y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}})$

f) $y^2.\sin(xy) = \tan x$ $\qquad (ans.: \frac{\sec^2 x - y^3.\cos(xy)}{2y.\sin(xy) + xy^2.\cos(xy)})$

g) $\sinh y = \tan^2 x$ $\qquad (ans.: \frac{2.\tan x.\sec^2 x}{\cosh y})$

10. Prove the following formulas :

a) $\frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$

b) $\frac{d}{dx}\csc u = -\csc u.\cot u.\frac{du}{dx}$

c) $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1 - u^2}}.\frac{du}{dx}$

d) $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}}.\frac{du}{dx}$

e) $\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$

f) $\frac{d}{dx}\csc h\ u = -\csc h\ u.\coth u.\frac{du}{dx}$

g) $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}}.\frac{du}{dx}$

h) $\frac{d}{dx}\sec h^{-1} u = -\frac{1}{|u|\sqrt{1 - u^2}}.\frac{du}{dx}$

11. Show that the tangent to the hyperbola $x^2 - y^2 = 1$ **at the point** $P(\cosh u, \sinh u)$ **, cuts the x-axis at the point** $(\operatorname{sech} u, 0)$ **and except when vertical , cuts the y-axis at the point** $(0, -\operatorname{csch} u)$ **.**