Chapter five

Integration

5-1- Indefinite integrals :

The set of all anti derivatives of a function is called indefinite integral of the function.

Assume *u* and *v* denote differentiable functions of *x*, and *a*, *n*, and *c* are constants, then the integration formulas are:-

1) $\int du = u(x) + c$

2) $\int a \cdot u(x)dx = a\int u(x)dx$

3) $\int (u(x) \mp v(x))dx = \int u(x)\,dx \mp \int v(x)\,dx$

4) $\int u^n du = \dfrac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1} du = \int \dfrac{1}{u} du = \ln u + c$

5) $\int a^u du = \dfrac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$

EX-1 **– Evaluate the following integrals:**

1) $\int 3x^2\,dx$

2) $\int \left(\dfrac{1}{x^2} + x\right)dx$

3) $\int x\sqrt{x^2+1}\,dx$

4) $\int (2t + t^{-1})^2\,dt$

5) $\int \sqrt{(z^2 - z^{-2})^2 + 4}\,dz$

6) $\int \dfrac{x+3}{\sqrt{x^2+6x}}\,dx$

7) $\int \dfrac{x+2}{x^2}\,dx$

8) $\int \dfrac{e^x}{1+3e^x}\,dx$

9) $\int 3x^3 \cdot e^{-2x^4}\,dx$

10) $\int 2^{-4x}\,dx$

Sol. **–**

1) $\int 3x^2\,dx = 3\int x^2 dx = 3\dfrac{x^3}{3} + c = x^3 + c$

$$2)\ \left(x^{-2}+x\right)dx=\int x^{-2}\,dx+\int x\,dx=\frac{x^{-1}}{-1}+\frac{x^{2}}{2}+c=-\frac{1}{x}+\frac{x^{2}}{2}+c$$

$$3)\ \int x\sqrt{x^{2}+1}\,dx=\frac{1}{2}\int 2x(x^{2}+1)^{1/2}dx=\frac{1}{2}\frac{(x^{2}+1)^{3/2}}{3/2}+c=\frac{1}{3}\sqrt{(x^{2}+1)3}+c$$

$$4)\ \int\left(2t+t^{-1}\right)^{2}dt=\int\left(4t^{2}+4+t^{-2}\right)dt=4\frac{t^{3}}{3}+4t+\frac{t^{-1}}{-1}+c=\frac{4}{3}t^{3}+4t-\frac{1}{t}+c$$

$$5)\ \int\sqrt{(z^{2}-z^{-2})^{2}+4}\,dz=\int\sqrt{z^{4}-2+z^{-4}+4}\,dz=\int\sqrt{z^{4}+2+z^{-4}}\,dz$$

$$=\int\sqrt{(z^{2}+z^{-2})^{2}}\,dz=\int(z^{2}+z^{-2})\,dz=\frac{z^{3}}{3}+\frac{z^{-1}}{-1}+c=\frac{1}{3}z^{3}-\frac{1}{z}+c$$

$$6)\ \int\frac{x+3}{\sqrt{x^{2}+6x}}\,dx=\frac{1}{2}\int(2x+6)\cdot(x^{2}+6x)^{-1/2}dx$$

$$=\frac{1}{2}\cdot\frac{(x^{2}+6x)^{1/2}}{1/2}+c=\sqrt{x^{2}+6x}+c$$

$$7)\ \int\frac{x+2}{x^{2}}\,dx=\int\left(\frac{x}{x^{2}}+\frac{2}{x^{2}}\right)dx=\int\left(x^{-1}+2x^{-2}\right)dx=\ln x+\frac{2x^{-1}}{-1}+c=\ln x-\frac{2}{x}+c$$

$$8)\ \int\frac{e^{x}}{1+3e^{x}}\,dx=\frac{1}{3}\int 3e^{x}(1+3e^{x})^{-1}dx=\frac{1}{3}\ln(1+3e^{x})+c$$

$$9)\ \int 3x^{3}\cdot e^{-2x^{4}}dx=-\frac{3}{8}\int -8x^{3}\cdot e^{-2x^{4}}dx=-\frac{3}{8}\cdot e^{-2x^{4}}+c$$

$$10)\ \int 2^{-4x}dx=-\frac{1}{4}\int 2^{-4x}\cdot(-4dx)=-\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2}+c$$

5-2- Integrals of trigonometric functions :

The integration formulas for the trigonometric functions are:

$$6)\ \int \sin u\cdot du=-\cos u+c$$

$$7)\ \int \cos u\cdot du=\sin u+c$$

$$8)\ \int \tan u\cdot du=-\ln|\cos u|+c$$

$$9)\ \int \cot u\cdot du=\ln|\sin u|+c$$

$$10)\ \int \sec u\cdot du=\ln|\sec u+\tan u|+c$$

$$11)\ \int \csc u\cdot du=-\ln|\csc u+\cot u|+c$$

$$12)\ \int \sec^{2}u\cdot du=\tan u+c$$

$$13)\ \int \csc^{2}u\cdot du=-\cot u+c$$

$$14)\ \int \sec u\cdot\tan u\cdot du=\sec u+c$$

$$15)\ \int \csc u\cdot\cot u\cdot du=-\csc u+c$$

EX-2- **Evaluate the following integrals:**

1) $\int \cos(3\theta-1)d\theta$

2) $\int x\cdot\sin(2x^2)\,dx$

3) $\int \cos^2(2y)\cdot\sin(2y)\,dy$

4) $\int \sec^3 x\cdot\tan x\;dx$

5) $\int \sqrt{2+\sin 3t}\cdot\cos 3t\;dt$

6) $\int \frac{d\theta}{\cos^2\theta}$

7) $\int (1-\sin^2 3t)\cdot\cos 3t\;dt$

8) $\int \tan^3(5x)\cdot\sec^2(5x)\;dx$

9) $\int \sin^4 x\cdot\cos^3 x\;dx$

10) $\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}dx$

Sol.-

1) $\frac{1}{3}\int 3\cos(3\theta-1)d\theta=\frac{1}{3}\sin(3\theta-1)+c$

2) $\frac{1}{4}\int 4x\cdot\sin(2x^2)dx=-\frac{1}{4}\cos(2x^2)+c$

3) $-\frac{1}{2}\int(\cos 2y)^2\cdot(-2\sin 2y\;dy)=-\frac{1}{2}\cdot\frac{(\cos 2y)^3}{3}+c=-\frac{1}{6}(\cos 2y)^3+c$

4) $\int \sec^2 x\cdot(\sec x\cdot\tan x\cdot dx)=\frac{\sec^3 x}{3}+c$

5) $\frac{1}{3}\int(2+\sin 3t)^{1/2}(3\cos 3t\;dt)=\frac{1}{3}\cdot\frac{(2+\sin 3t)^{3/2}}{3/2}+c=\frac{2}{9}\sqrt{(2+\sin 3t)^3}+c$

6) $\int\frac{d\theta}{\cos^2\theta}=\int\sec^2\theta\cdot d\theta=\tan\theta+c$

7) $\int(1-\sin^2 3t)\cdot\cos 3t\;dt=\frac{1}{3}\int 3\cos 3t\;dt-\frac{1}{3}\int(\sin 3t)^2\cdot 3\cos 3t\;dt$

$=\frac{1}{3}\sin 3t-\frac{1}{3}\cdot\frac{\sin^3 3t}{3}+c=\frac{1}{3}\cdot\sin 3t-\frac{1}{9}\sin^3 3t+c$

8) $\frac{1}{5}\int\tan^3 5x\cdot(5\sec^2 5x\;dx)=\frac{1}{5}\cdot\frac{\tan^4 5x}{4}+c=\frac{1}{20}\tan^4 5x+c$

9) $\int\sin^4 x\cdot\cos^3 x\;dx=\int\sin^4 x\cdot(1-\sin^2 x)\cdot\cos x\;dx$

$=\int\sin^4 x\cdot\cos x\;dx-\int\sin^6 x\cdot\cos x\;dx=\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

5-3- Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$16)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$$

$$17)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$18)\ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \quad ; \quad \forall u^2 > a^2$$

EX-3 **Evaluate the following integrals:**

$1)\ \int \frac{x^2}{\sqrt{1-x^6}}\,dx$

$2)\ \int \frac{dx}{\sqrt{9-x^2}}$

$3)\ \int \frac{x}{1+x^4}\,dx$

$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$5)\ \int \frac{dx}{x\sqrt{4x^2-1}}$

$6)\ \int \frac{2dx}{\sqrt{x}(1+x)}$

$7)\ \int \frac{dx}{1+3x^2}$

$8)\ \int \frac{2\cos x}{1+\sin^2 x}\,dx$

$9)\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$10)\ \int \frac{\tan^{-1}x}{1+x^2}\,dx$

Sol.-

$$1)\ \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right) = \frac{1}{3}\sin^{-1}x^3 + c$$

$$2)\ \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$$

$$3)\ \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1} x^2 + c$$

$$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$$

$$5)\ \int \frac{2\ dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2\sqrt{x}}\ dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x} + c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\ dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c$$

$$8)\ 2\int \frac{\cos x\ dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x) + c$$

$$9)\ \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10)\ \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19)\ \int \sinh u \cdot du = \cosh u + c$$

$$20)\ \int \cosh u \cdot du = \sinh u + c$$

$$21)\ \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22)\ \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23)\ \int \operatorname{sech}^2 u \cdot du = \tanh u + c$$

$$24)\ \int \operatorname{csch}^2 u \cdot du = \coth u + c$$

$$25)\ \int \operatorname{sech} u \cdot \tanh u \cdot du = -\operatorname{sech} u + c$$

$$26)\ \int \operatorname{csch} u \cdot \coth u \cdot du = -\operatorname{csch} u + c$$

EX-4 – **Evaluate the following integrals:**

1) $\int \frac{cosh(lnx)}{x} dx$

2) $\int sinh(2x+1)\,dx$

3) $\int \frac{sinhx}{cosh^4 x} dx$

4) $\int x \cdot cosh(3x^2)\,dx$

5) $\int sinh^4 x \cdot cosh\,x\ dx$

6) $\int sech^2(2x-3)\,dx$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

8) $\int \left(e^{ax} - e^{-ax}\right) dx$

9) $\int \frac{sinh\,x}{1+cosh\,x} dx$

10) $\int csch^2 x \cdot cothx\,dx$

Sol.-

1) $\int cosh(ln\,x) \cdot \left(\frac{dx}{x}\right) = sinh(ln\,x) + c$

2) $\frac{1}{2}\int sinh(2x+1)\cdot(2\,dx) = \frac{1}{2}cosh(2x+1) + c$

3) $\int \frac{1}{cosh^3 x} \cdot \frac{sinh\,x}{cosh\,x}\ dx = \int sech^3 x \cdot tanh\,x\ dx$

$= -\int sech^2 x \cdot \left(-sech\,x \cdot tanh\,x\ dx\right) = -\frac{sech^3 x}{3} + c$

4) $\frac{1}{6}\int cosh(3x^2)\cdot(6x\,dx) = \frac{1}{6}sinh(3x^2) + c$

5) $\int sinh^4 x \cdot \left(cosh\,x\ dx\right) = \frac{sinh^5 x}{5} + c$

6) $\frac{1}{2}\int sech^2(2x-3)\cdot(2\ dx) = \frac{1}{2}tanh(2x-3) + c$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int tanh\,x\ dx = ln(cosh\,x) + c$

8) $2\int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a}\int sinh\,ax\ (a\,dx) = \frac{2}{a}cosh\,ax + c$

9) $\int \frac{sinh\,x\ dx}{1+cosh\,x} = ln(1+cosh\,x) + c$

10) $-\int csch\,x \cdot \left(-csch\,x \cdot coth\,x\ dx\right) = -\frac{csch^2 x}{2} + c$

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27)\ \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28)\ \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29)\ \int \frac{du}{1-u^2} = \begin{Bmatrix} \tanh^{-1} u + c & if \ |u|<1 \\ \coth^{-1} u + c & if \ |u|>1 \end{Bmatrix} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c$$

$$30)\ \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31)\ \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

EX-4 **– Evaluate the following integrals:**

$1)\ \int \frac{dx}{\sqrt{1+4x^2}}$ $\quad 2)\ \int \frac{dx}{\sqrt{4+x^2}}$ $\quad 3)\ \int \frac{dx}{1-x^2}$

$4)\ \int \frac{dx}{x\sqrt{4+x^2}}$ $\quad 5)\ \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta-1}}$ $\quad 6)\ \int \tanh^{-1}\left(\ln\sqrt{x}\right)\cdot\frac{dx}{x\left(1-\ln^2\sqrt{x}\right)}$

Sol.-

$$1)\ \frac{1}{2}\int \frac{2\ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$$

$$2)\ \int \frac{\frac{1}{2}\ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$$

$$3)\ \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \qquad if \quad |x|<1$$

$$= \coth^{-1} x + c \qquad if \quad |x|>1$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{\frac{1}{2}\,dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2}\operatorname{csc}h^{-1}\left|\frac{x}{2}\right| + c$$

$$5)\ \int \frac{1}{\sqrt{\tan^2\theta - 1}}\left(\sec^2\theta\ d\theta\right) = \cosh^{-1}(\tan\theta) + c$$

$$6)\quad let\quad u = \ln\sqrt{x} = \frac{1}{2}\ln x \qquad du = \frac{1}{2x}dx$$

$$\int \tanh^{-1}(\ln\sqrt{x})\cdot\frac{dx}{x(1-\ln^2\sqrt{x})} = \int \tanh^{-1}u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\frac{\left(\tanh^{-1}u\right)^2}{2} + c = \left[\tanh^{-1}(\ln\sqrt{x})\right]^2 + c$$

Problems – 5

Evaluate the following integrals:

1) $\int (x^2-1)\cdot(4-x^2)\,dx$ (ans.: $\frac{5}{3}x^3-\frac{1}{5}x^5-4x+c$)

2) $\int e^x\cdot \sin e^x\,dx$ (ans.: $-\cos e^x+c$)

3) $\int \tan(3x+5)\,dx$ (ans.: $-\frac{1}{3}\ln|\cos(3x+5)|+c$)

4) $\int \frac{\cot(\ln x)}{x}dx$ (ans.: $\ln|\sin(\ln x)|+c$)

5) $\int \frac{\sin x+\cos x}{\cos x}dx$ (ans.: $-\ln|\cos x|+x+c$)

6) $\int \frac{dx}{1+\cos x}$ (ans.: $-\cot x+\csc x+c$)

7) $\int \cot(2x+1)\cdot\csc^2(2x+1)\,dx$ (ans.: $-\frac{1}{4}\cot^2(2x+1)+c$)

8) $\int \frac{dx}{\sqrt{1-9x^2}}$ (ans.: $\frac{1}{3}\sin^{-1}(3x)+c$)

9) $\int \frac{dx}{\sqrt{2-x^2}}$ (ans.: $\sin^{-1}\frac{x}{\sqrt{2}}+c$)

10) $\int e^{2x}\cdot \cosh e^{2x}\,dx$ (ans.: $\frac{1}{2}\sinh e^{2x}+c$)

11) $\int e^{\sin x}\cdot\cos x\,dx$ (ans.: $e^{\sin x}+c$)

12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3}e^{-3x}+c$)

13) $\int \frac{e^{\sqrt{x}}-1}{\sqrt{x}}dx$ (ans.: $2e^{\sqrt{x}}-2\sqrt{x}+c$)

14) $\int x\left(a+b\sqrt{3x}\right)dx$ where a,b constants (ans.: $\frac{1}{10}(5ax^2+4\sqrt{3}bx^{5/2})+c$)

15) $\int \frac{dx}{-1-x^2}$ (ans.: $-\tan^{-1}x+c$)

16) $\int \frac{\cos\theta\,d\theta}{1+\sin^2\theta}$ (ans.: $\tan^{-1}(\sin\theta)+c$)

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17) $\int \frac{1}{x^2} \csc\frac{1}{x} \cot\frac{1}{x}\, dx$ (ans.: $\csc\frac{1}{x} + c$)

18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}\, dx$ (ans.: $\frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2} + c$)

19) $\int \sin(\tan\theta)\cdot\sec^2\theta\, d\theta$ (ans.: $-\cos(\tan\theta) + c$)

20) $\int \sqrt{x^2 - x^4}\, dx$ (ans.: $-\frac{1}{3}\sqrt{(1-x^2)^3} + c$)

21) $\int \frac{\sec^2 2x\ dx}{\sqrt{\tan 2x}}$ (ans.: $\sqrt{\tan 2x} + c$)

22) $\int (\sin\theta - \cos\theta)^2\, d\theta$ (ans.: $\theta + \cos^2\theta + c$)

23) $\int \frac{y}{y^4+1}\, dy$ (ans.: $\frac{1}{2}\tan^{-1} y^2 + c$)

24) $\int \frac{dx}{\sqrt{x}(x+1)}$ (ans.: $2\tan^{-1}\sqrt{x} + c$)

25) $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}}\, dt$ (ans.: $\frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}} + c$)

26) $\int \frac{dx}{x^{\frac{1}{5}}\sqrt{1+x^{\frac{4}{5}}}}$ (ans.: $\frac{5}{2}\sqrt{1+x^{\frac{4}{5}}} + c$)

27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}\, dx$ (ans.: $-\frac{1}{12}(\cos^{-1} 4x)^3 + c$)

28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ (ans.: $\sec^{-1}(2x) + c$)

29) $\int \frac{dx}{(e^x + e^{-x})^2}$ (ans.: $\frac{1}{4}\tanh x + c$)

30) $\int 3^{\ln x^2}\, \frac{dx}{x}$ (ans.: $\frac{1}{2\ln 3} 3^{\ln x^2} + c$)

31) $\int \frac{\cot x\ dx}{\ln(\sin x)}$ (ans.: $\ln\ln(\sin x) + c$)

32) $\int \frac{(\ln x)^2}{x}\, dx$ (ans.: $\frac{1}{3}(\ln x)^3 + c$)

33) $\int \frac{\sin x\cdot e^{\sec x}}{\cos^2 x}\, dx$ (ans.: $e^{\sec x} + c$)

34) $\int \frac{dx}{x \cdot \ln x}$ (ans.: $\ln \ln x + c$)

35) $\int \frac{d\theta}{\cosh\theta + \sinh\theta}$ (ans.: $-e^{-\theta} + c$)

36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}}\, dx$ (ans.: $x - \frac{1}{5\ln 2} 2^{5x} + c$)

37) $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2}\, dt$ (ans.: $\frac{1}{2} e^{\tan^{-1} 2t} + c$)

38) $\int \frac{\cot x}{\csc x}\, dx$ (ans.: $\sin x + c$)

39) $\int \sec^4 x \cdot \tan^3 x \; dx$ (ans.: $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c$)

40) $\int \csc^4 3x \; dx$ (ans.: $-\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c$)

41) $\int \frac{\cos^3 t}{\sin^2 t}\, dt$ (ans.: $-\csc t - \sin t + c$)

42) $\int \frac{\sec^4 x}{\tan^4 x}\, dx$ (ans.: $-\frac{1}{3}\cot^3 x - \cot x + c$)

43) $\int \tan^2 4\theta \; d\theta$ (ans.: $\frac{1}{4}\tan 4\theta - \theta + c$)

44) $\int \frac{e^x}{1 + e^x}\, dx$ (ans.: $\ln(1 + e^x) + c$)

45) $\int \tan^3 2x \; dx$ (ans.: $\frac{1}{4}\tan^2 2x + \frac{1}{2}\ln|\cos 2x| + c$)

46) $\int \frac{\sec^2 x}{2 + \tan x}\, dx$ (ans.: $\ln(2 + \tan x) + c$)

47) $\int \sec^4 3x \; dx$ (ans.: $\frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + c$)

48) $\int \frac{e^t}{1 + e^{2t}}\, dt$ (ans.: $\tan^{-1} e^t + c$)

49) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$ (ans.: $2\sin\sqrt{x} + c$)

50) $\int \frac{dx}{\sin x \cdot \cos x}$ (ans.: $-\ln|\csc 2x + \cot 2x| + c$)

51) $\int \sqrt{1+\sin y}\; dy$ (ans.: $-2\sqrt{1-\sin y}+c$)

52) $\int \dfrac{dx}{(x^2+1)(2+\tan^{-1}x)}$ (ans.: $\ln(2+\tan^{-1}x)+c$)

53) $\int \sin^{-1}(\cosh x)\cdot\dfrac{\sinh x\; dx}{\sqrt{1-\cosh^2 x}}$ (ans.: $\dfrac{1}{2}\left(\sinh^{-1}(\cosh x)\right)^2+c$)

54) $\int \dfrac{\cos\theta\; d\theta}{1-\sin^2\theta}$ (ans.: $\ln\left|\sec\theta+\tan\theta\right|+c$)

55) $\int \dfrac{dx}{x\left(1+(\ln x)^2\right)}$ (ans.: $\tan^{-1}(\ln x)+c$)

56) $\int \left(e^{\frac{9}{4}x}-2e^{\frac{5}{4}x}+e^{\frac{x}{4}}\right)dx$ (ans.: $\dfrac{4}{9}e^{\frac{9}{4}x}-\dfrac{8}{5}e^{\frac{5}{4}x}+4e^{\frac{x}{4}}+c$)

57) $\int \dfrac{e^x\,dx}{e^{2x}+2e^x+1}$ (ans.: $-\dfrac{1}{e^x+1}+c$)

58) $\int e^x\cdot\sinh 2x\; dx$ (ans.: $\dfrac{1}{2}\left[\dfrac{1}{3}e^{3x}+e^{-x}\right]+c$)

59) $\int \dfrac{\sec^3 x+e^{\sin x}}{\sec x}dx$ (ans.: $\tan x+e^{\sin x}+c$)

60) $\int \dfrac{3^{x+2}}{2+9^{x+1}}dx$ (ans.: $\dfrac{3}{\sqrt{2}\ln 3}\tan^{-1}\dfrac{3^{x+1}}{\sqrt{2}}+c$)

61) $\int \dfrac{\cos x\; dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$ (ans.: $2\sin^{-1}\sqrt{\sin x}+c$)

62) $\int \tan^5 x\; dx$ (ans.: $\dfrac{1}{4}\sec^4 x-\sec^2 x-\ln\left|\cos x\right|+c$)

63) $\int e^{\ln\sin^{-1}x}\cdot\dfrac{dx}{\sqrt{1-x}^{2}}$ (ans.: $\dfrac{1}{2}(\sin^{-1}x)^2+c$)

64) $\int x\cdot e^{x^2-1}\; dx$ (ans.: $\dfrac{1}{2}e^{x^2-1}+c$)

65) $\int \cosh(\ln\cos x)\; dx$ (ans.: $\dfrac{1}{2}\left[\sin x+\ln\left|\sec x+\tan x\right|\right]+c$)

66) $\int \dfrac{\cos x}{\sin^2 x}dx$ (ans.: $-\csc x+c$)

67) $\int \cosh^{-1}(\sin x)\;\dfrac{\cos x\; dx}{\sqrt{\sin^2 x-1}}$ (ans.: $\dfrac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2+c$)