# Discrete Mathematics

## Lecture 2

Applications of Propositional Logic,
Propositional Equivalences

By

***Asst. Lect. Ali Al-Khawaja***

# Applications of Propositional Logic

1. Translating English Sentences.
2. System Specifications.
3. Boolean Searches.
4. Logic Puzzles.
5. Logic Circuits.

# Applications of Propositional Logic

1. Translating English Sentences.
2. System Specifications.
3. Boolean Searches.
4. Logic Puzzles.
5. Logic Circuits.

# Applications of Propositional Logic

## 1. Translating English Sentences

- There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into compound statements removes the ambiguity.

# Applications of Propositional Logic

## Example 1

You can access the Internet from campus only if you are a computer science major or you are not a student.

*Solution:*

Let $p, q$ and $r$ be the propositions:

$p$: You can access the Internet from campus.
$q$: You are a computer science major.
$r$: You are a student.

## Example 1

(You can access the Internet from campus) only if (you are a computer science major or you are not a student).

*Solution:*

Let $p, q$ and $r$ be the propositions:

$p \rightarrow q$
"$p$ only if $q$"

$p$: You can access the Internet from campus.
$q$: You are a computer science major.
$r$: You are a student.

## Example 1

(You can access the Internet from campus) only if (you are a computer science major or you are not a student).

*Solution:*

Let $p, q$ and $r$ be the propositions:

$p \to q$
"$p$ only if $q$"

$p$: You can access the Internet from campus.
$q$: You are a computer science major.
$r$: You are a student.

***<u>The sentence can be represented by logic as</u>*** $\boxed{p \to (q \vee \neg r)}$

# Example 2

The automated reply cannot be sent when the file system is full.

# Example 2

The automated reply cannot be sent when the file system is full.

*Solution:*

Let $p$ and $q$ be the propositions:

$p$: The automated reply can be sent .
$q$: The file system is full.

$p \to q$
"$q$ when $p$"

# Example 2

(The automated reply cannot be sent) **when** (the file system is full.)

*Solution:*

Let $p$ and $q$ be the propositions:

$p \rightarrow q$
"$q$ when $p$"

$p$: The automated reply can be sent .
$q$: The file system is full.

***The sentence can be represented by logic as***

$$\boxed{q \rightarrow \neg p}$$

# Applications of Propositional Logic

## 2. Logic Circuits

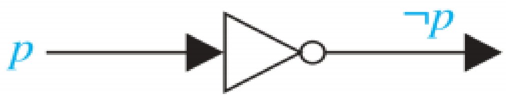
- A logic circuit (or digital circuit) receives input signals $p_1, p_2, \ldots, p_n$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s_1, s_2, \ldots, s_n$, each a bit.
- In this course, we will restrict our attention to logic circuits with a single output signal; in general, digital circuits may have multiple outputs.

# Applications of Propositional Logic

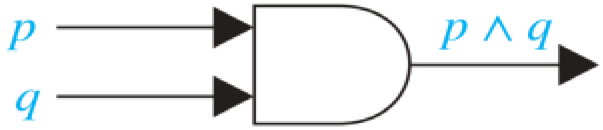
## 2. Logic Circuits

- Complicated digital circuits can be constructed from three basic circuits, called gates.


OR gate



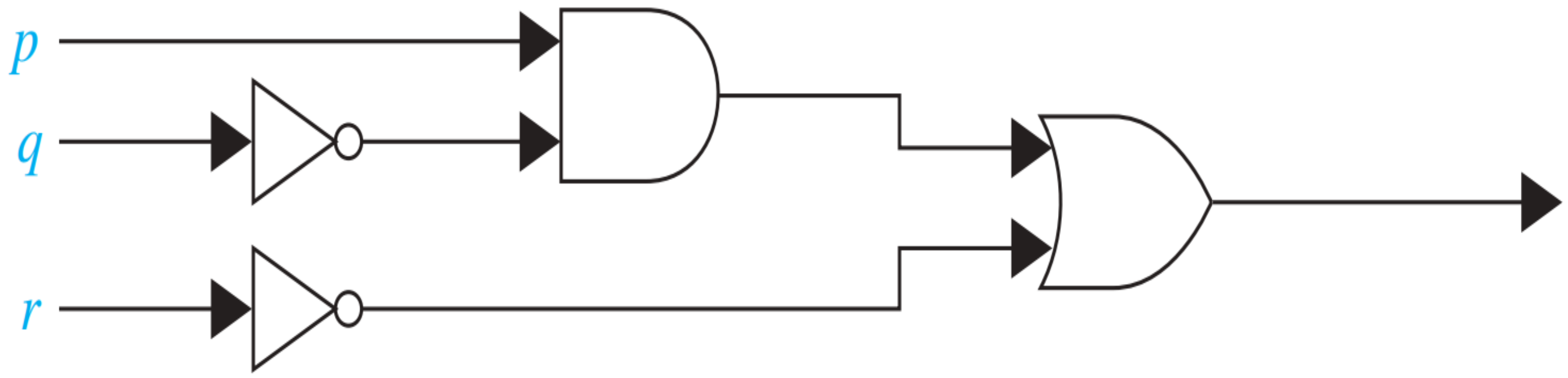

Inverter




AND gate

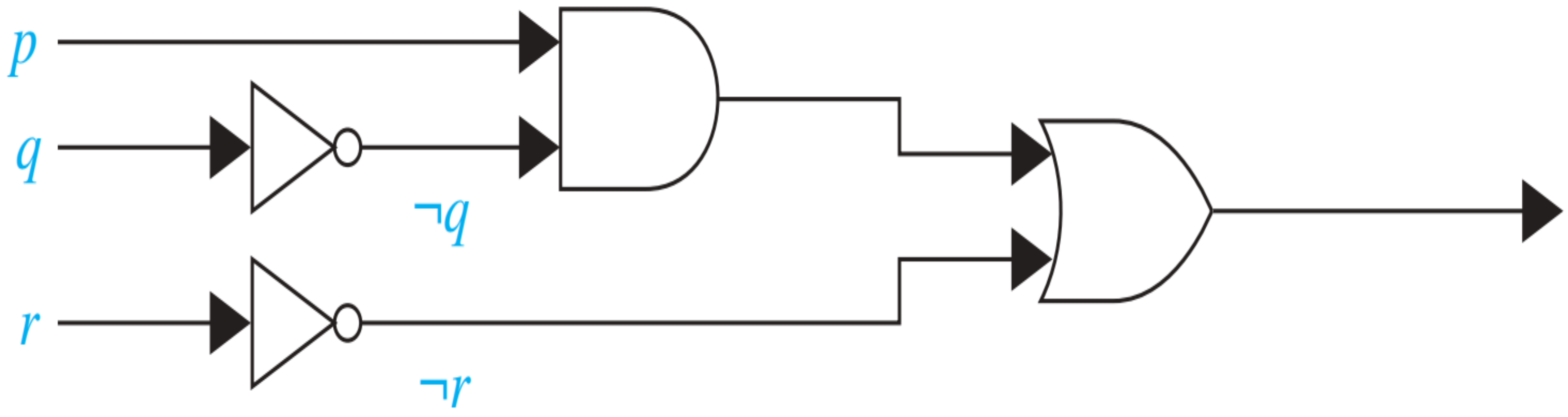
# Example 1

- Determine the output for the combinatorial circuit in the following figure.



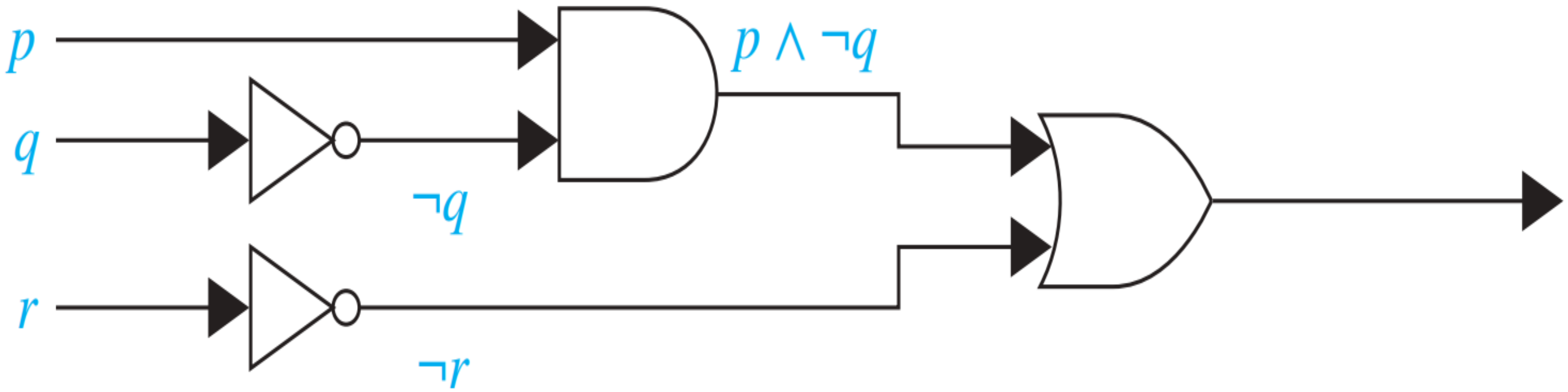
# Example 1

- Determine the output for the combinatorial circuit in the following figure.



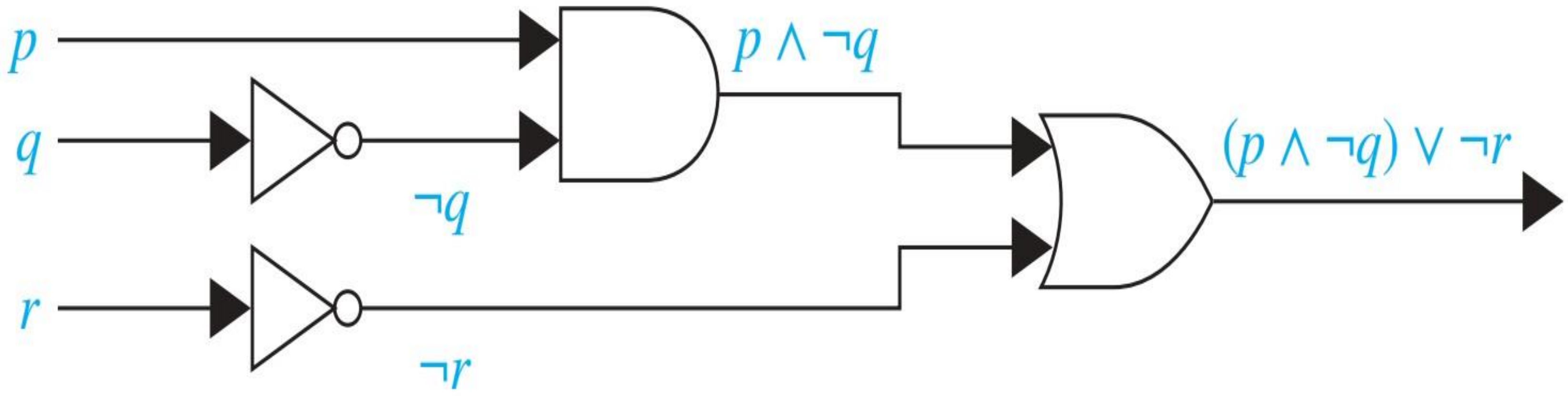
# Example 1

- Determine the output for the combinatorial circuit in the following figure.



# Example 1

- Determine the output for the combinatorial circuit in the following figure.



# Example 2

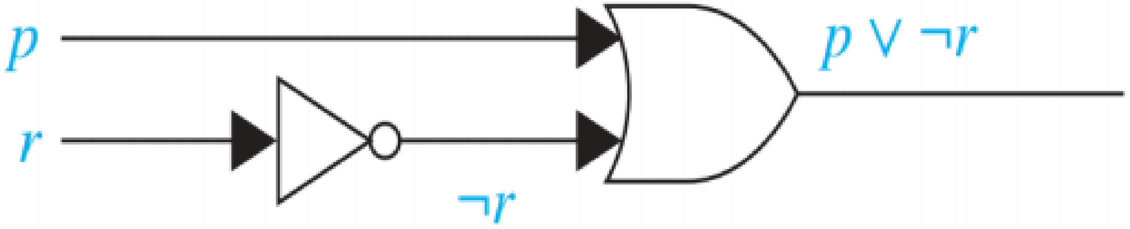
- Build a digital circuit that produces the output

$$(p \lor \neg r) \land (\neg p \lor (q \lor \neg r))$$

when given input bits $p$, $q$, and $r$.

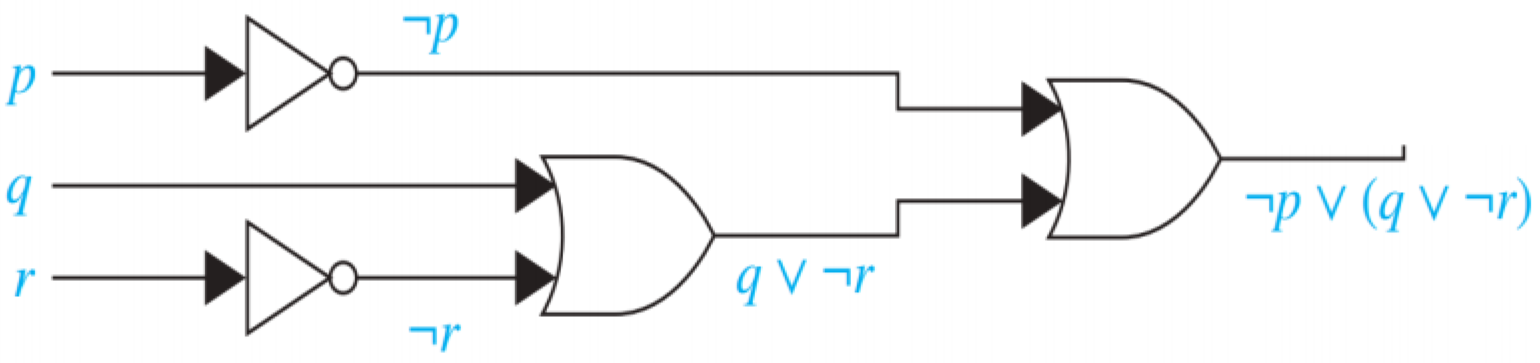
# Example 2

$$\boxed{(p \vee \neg r)} \wedge (\neg p \vee (q \vee \neg r))$$



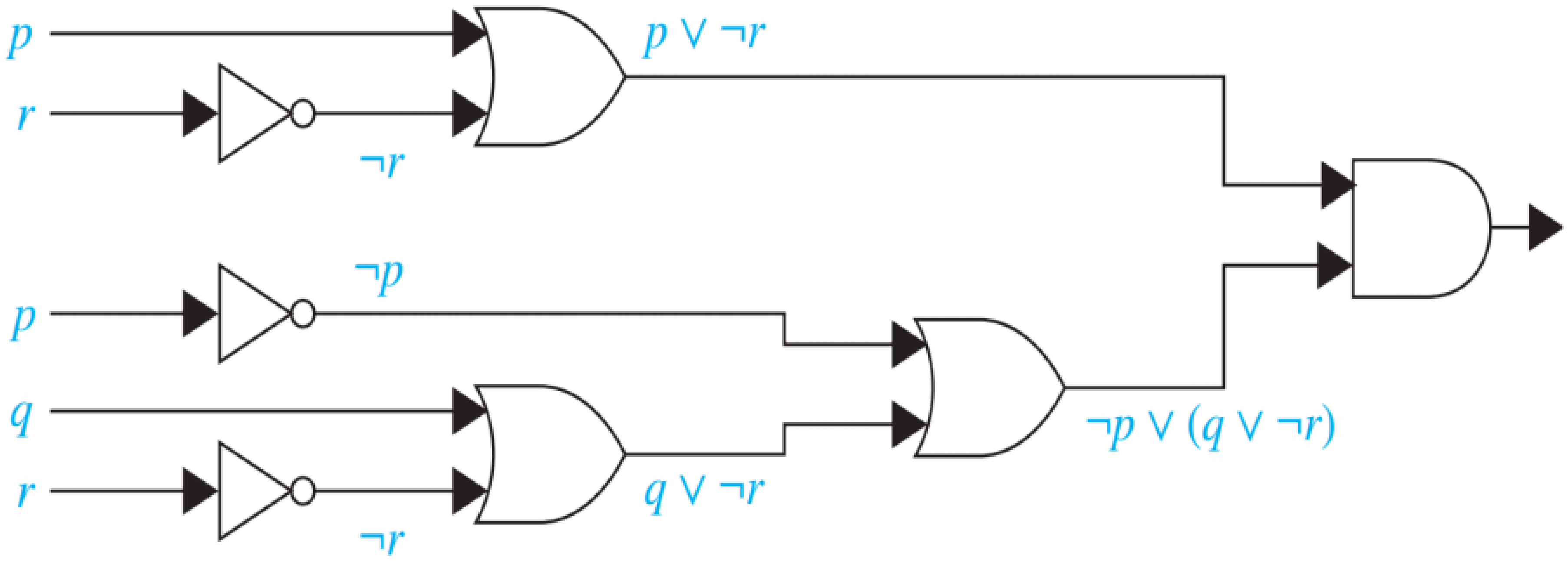
# Example 2

$$(p \vee \neg r) \wedge \boxed{(\neg p \vee (q \vee \neg r))}$$

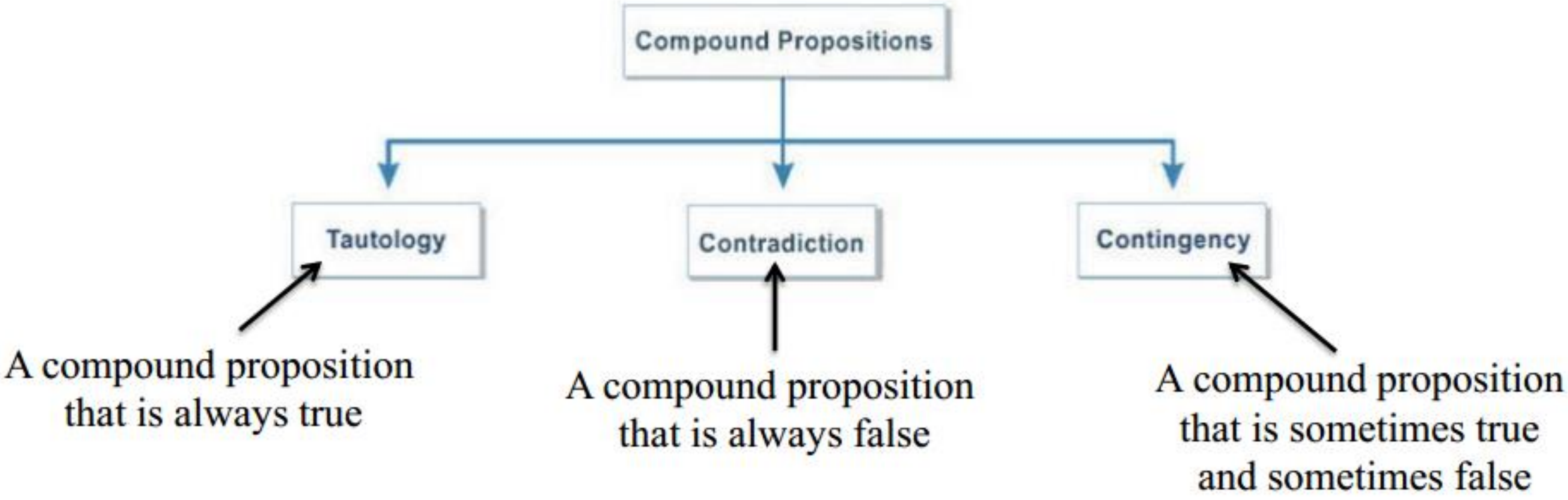


# Example 2

$$\boxed{(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))}$$



# Compound Propositions Classification



# Example

Show that following conditional statement is a **tautology** by using truth table.

$$(p \land q) \to p$$

| $p$ | $q$ | $p \land q$ | $(p \land q) \to p$ |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |

# Example

- Show that following conditional statement is a **tautology** by using truth table.

$$(p \wedge q) \to p$$

| $p$ | $q$ | $p \wedge q$ | $(p \wedge q) \to p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

# Logical Equivalences

## ❖ Logically equivalent

The compound propositions $p$ and $q$ are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that $p$ and $q$ are logically equivalent.

Compound propositions that have the **same truth values** in **all** possible cases are called **logically equivalent**.

$\equiv$

$\Leftrightarrow$

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | | | | |
| T | F | T | | | | |
| F | T | T | | | | |
| F | F | F | | | | |

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | | | |
| T | F | T | F | | | |
| F | T | T | F | | | |
| F | F | F | T | | | |

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | |
| T | F | T | F | F | T | |
| F | T | T | F | T | F | |
| F | F | F | T | T | T | |

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

# Example1

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

**Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.**

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

| **Logical Equivalences.** | |
|---|---|
| ***Equivalence*** | ***Name*** |
| $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity laws |
| $p \vee \mathbf{T} \equiv \mathbf{T}$<br>$p \wedge \mathbf{F} \equiv \mathbf{F}$ | Domination laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent laws |
| $\neg(\neg p) \equiv p$ | Double negation law |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |

| **Logical Equivalences.** | |
|---|---|
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee \neg p \equiv \mathbf{T}$<br>$p \wedge \neg p \equiv \mathbf{F}$ | Negation laws |

**Logical Equivalences Involving Conditional Statements.**

$p \to q \equiv \neg p \lor q$

$p \to q \equiv \neg q \to \neg p$

$p \lor q \equiv \neg p \to q$

$p \land q \equiv \neg(p \to \neg q)$

**Logical Equivalences Involving Biconditional Statements.**

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ by the second De Morgan law

| | |
|---|---|
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$\begin{aligned} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \end{aligned}$$

| | |
|---|---|
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}
\end{aligned}
$$

| | |
|---|---|
| $\neg(\neg p) \equiv p$ | Double negation law |

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

| | |
|---|---|
| $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ | by the second De Morgan law |
| $\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ | by the first De Morgan law |
| $\equiv \neg p \wedge (p \vee \neg q)$ | by the double negation law |
| $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ | by the second distributive law |

| | |
|---|---|
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}
\end{aligned}
$$

## Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

| | |
|---|---|
| $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ | by the second De Morgan law |
| $\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ | by the first De Morgan law |
| $\equiv \neg p \wedge (p \vee \neg q)$ | by the double negation law |
| $\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ | by the second distributive law |
| $\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$ | because $\neg p \wedge p \equiv \mathbf{F}$ |
| $\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$ | by the commutative law for disjunction |

| | |
|---|---|
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |

# Example 1:

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}$$

| | |
|---|---|
| $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity laws |

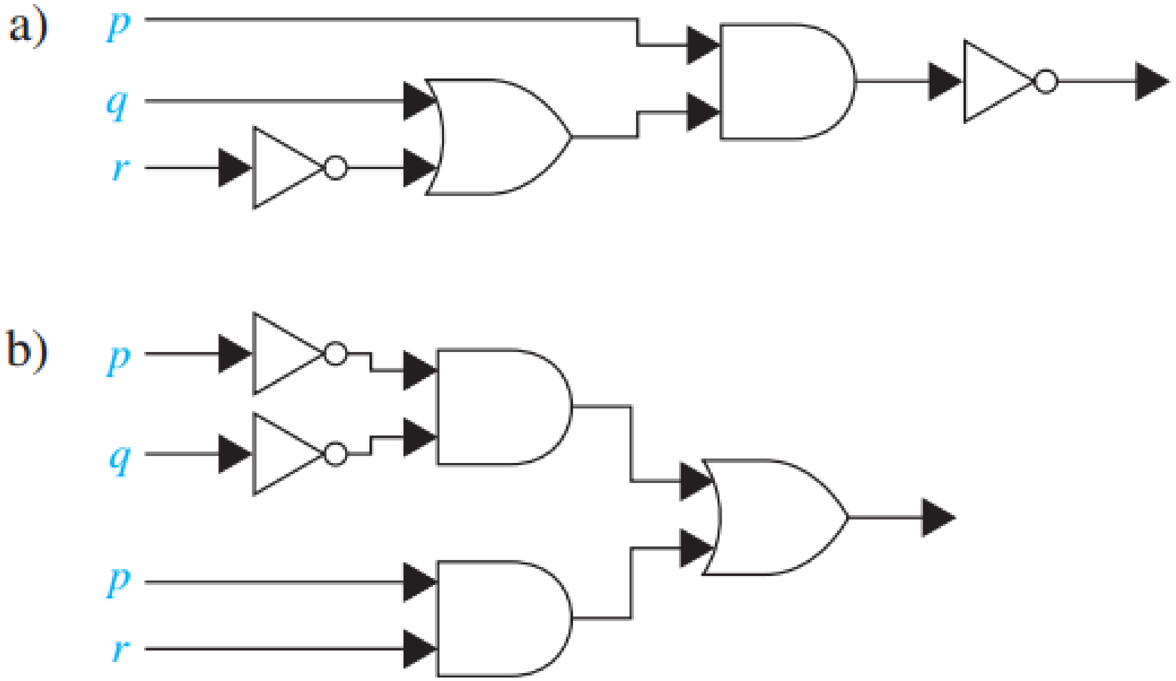
## Example 1:

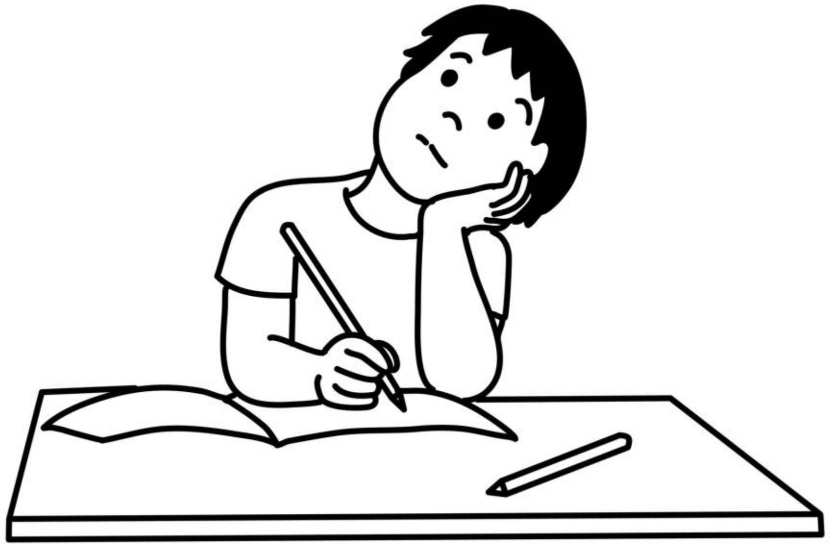
Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

$$
\begin{aligned}
\boxed{\neg(p \lor (\neg p \land q))} &\equiv \neg p \land \neg(\neg p \land q) && \text{by the second De Morgan law} \\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \land (p \lor \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \land p) \lor (\neg p \land \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \lor (\neg p \land \neg q) && \text{because } \neg p \land p \equiv \mathbf{F} \\
&\equiv (\neg p \land \neg q) \lor \mathbf{F} && \text{by the commutative law for disjunction} \\
&\boxed{\equiv \neg p \land \neg q} && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

# Homework 1

Find the output of each of these combinatorial circuits.



# Homework 2

Show that $(\mathbf{p} \vee \mathbf{q}) \wedge (\neg \mathbf{p} \vee \mathbf{r}) \rightarrow (\mathbf{q} \vee \mathbf{r})$ is a tautology.