ALMUSTAQBAL UNIVERSITY

COLLEGE OF ENGINEERING AND TECHNOLOGY

COMPUTER ENGINEERING TECHNIQUE DEPARTMENT

Subject :Laplace Transform

Lecture No five

Lecture title: Laplace Transform

Lecturer name: Laplace inverse

2. LAPLACE INVERSE TRANSFORM

2.1 Definition of Inverse Laplace Transformation:

If the Laplace Transform of $f(t)$ is $F(s)$, i.e. if $L\{f(t)\} = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ i.e.

$$\boxed{L^{-1}\{F(s)\} = f(t)}$$

where, L^{-1} is called the inverse Laplace transformation operator.

Inverse Laplace Transform of some elementary functions:

S. No.	$F(s)$	$L^{-1}\{F(s)\} = f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}}$; $n = 0, 1, 2...$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-1}$	e^t
5.	$\frac{1}{s^2+1}$	$\sin t$
6.	$\frac{s}{s^2+1}$	$\cos t$
7.	$\frac{1}{s^2-1}$	$\sinh t$
8.	$\frac{s}{s^2-1}$	$\cosh t$

2.3 Change of Scale Property:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

Proof: By definition, we have $L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \Rightarrow L^{-1}\{F(s)\} = f(t)$

$$\therefore \quad L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^\infty f\left(\frac{t}{a}\right) e^{-st}\, dt \qquad \text{Let } \frac{t}{a} = u \quad \therefore \quad dt = a\, du$$

$$\Rightarrow \quad L\left\{f\left(\frac{t}{a}\right)\right\} = \int_0^\infty f(u)\, e^{-sau}\, a\, du = a\, F(as)$$

Taking Inverse Laplace Transform on both sides, we get

$$\Rightarrow \qquad f\left(\frac{t}{a}\right) = L^{-1}[a\,F(a\,s)]$$

i. e., $$L^{-1}[F(a\,s)] = \frac{1}{a}\,f\left(\frac{t}{a}\right)$$

Similarly, we can prove that

$$L^{-1}[F(s/a)] = a\,f(t\,a)$$

Examples:

1. If $L^{-1}\left\{\frac{1}{s^2+64}\right\} = \frac{\sin 8t}{8}$, find $L^{-1}\left\{\frac{1}{s^2+4}\right\}$ where $a > 0$

Sol. Given $L^{-1}\left\{\frac{1}{s^2+64}\right\} = \frac{\sin 8t}{8}$

$$\therefore \qquad L^{-1}\left\{\frac{1}{(4s)^2+64}\right\} = \frac{1}{4}\,\frac{\sin\,(8t/4)}{8} \qquad \left[\because\ L^{-1}\{F(a\,s)\} = (1/a)\,f\,(t/a)\right]$$

$$\Rightarrow \qquad \frac{1}{16}\,L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{16}\,\frac{\sin 2t}{2}$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{\sin 2t}{2}$$

2. If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi\,t}}$, find $L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$ where $a > 0$

Sol. Given $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi\,t}}$

$$\therefore \qquad L^{-1}\left\{\frac{e^{-a/s}}{(s/a)^{1/2}}\right\} = a\,\frac{\cos 2\sqrt{t\,a}}{\sqrt{\pi\,t\,a}} \qquad \left[\because\ L^{-1}\{F\,(s/a)\} = a\,f\,(t\,a)\right]$$

$$\Rightarrow \qquad \sqrt{a}\;L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \sqrt{a}\;\frac{\cos 2\sqrt{t\,a}}{\sqrt{\pi\,t}}$$

$$\Rightarrow \qquad L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t\,a}}{\sqrt{\pi\,t}}$$

2.4 Linearity Property:

If C_1 and C_2 are any constants, while $F_1(s)$ and $F_2(s)$ are Laplace Transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}\{C_1\,F_1(s) + C_2\,F_2(s)\} = C_1\;L^{-1}\{F_1(s)\} + C_2\;L^{-1}\{F_2(s)\} = C_1\,f_1(t) + C_2\,f_2(t)$$

Proof: By definition, we have $L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = F(s) \quad \Rightarrow \quad L^{-1}\{F(s)\} = f(t)$

Now, $L\{C_1 f_1(t) + C_2 f_2(t)\} = \int_0^\infty [C_1 f_1(t) + C_2 f_2(t)] e^{-st}\, dt$

$$= C_1 \int_0^\infty f_1(t)\, e^{-st}\, dt + C_2 \int_0^\infty f_2(t)\, e^{-st}\, dt$$

$$= C_1 L\{f_1(t)\} + C_2 L\{f_2(t)\}$$

$$= C_1 F_1(s) + C_2 F_2(s)$$

Taking Inverse Laplace Transform on both sides, we get

$\Rightarrow \quad L^{-1}\{C_1 F_1(s) + C_2 F_2(s)\} = C_1\ f_1(t) + C_2\ f_2(t)$

$\Rightarrow \quad L^{-1}\{C_1 F_1(s) + C_2 F_2(s)\} = C_1\ L^{-1}\{F_1(s)\} + C_2\ L^{-1}\{F_2(s)\}$

Examples:

1. Find $L^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s-18}{s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}\right\}$

Sol. $L^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s-18}{s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}\right\}$

$$= L^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+3^2} + \frac{18}{s^2+3^2} + \frac{24}{s^4} - \frac{30}{s^{7/2}}\right\}$$

$$= L^{-1}\left\{\frac{5}{\Gamma(2)}\frac{\Gamma(2)}{s^2} + \frac{4}{\Gamma(3)}\frac{\Gamma(3)}{s^3} - \frac{2s}{s^2+3^2} + \frac{6\times 3}{s^2+3^2} + \frac{24}{\Gamma(4)}\frac{\Gamma(4)}{s^4} - \frac{30}{\Gamma(7/2)}\frac{\Gamma(7/2)}{s^{7/2}}\right\}$$

$$= \frac{5}{\Gamma(2)} L^{-1}\left\{\frac{\Gamma(2)}{s^2}\right\} + \frac{4}{\Gamma(3)} L^{-1}\left\{\frac{\Gamma(3)}{s^3}\right\} - 2L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 6L^{-1}\left\{\frac{3}{s^2+3^2}\right\} + \frac{24}{\Gamma(4)} L^{-1}\left\{\frac{\Gamma(4)}{s^4}\right\} - \frac{30}{\Gamma(7/2)} L^{-1}\left\{\frac{\Gamma(7/2)}{s^{7/2}}\right\}$$

$$= \frac{5}{1!}t + \frac{4}{2!}t^2 - 2\cos 3t + 6\sin 3t + \frac{24}{3!}t^3 - \frac{30}{(5/2)(3/2)(1/2)\sqrt{\pi}}t^{5/2}$$

$$\left[\because\ L^{-1}\left\{\frac{\Gamma(n)}{s^n}\right\} = t^{n-1},\ L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,\ L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at\right]$$

$$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - \frac{16}{\sqrt{\pi}}t^{5/2}$$

2. Find $L^{-1}\left\{\dfrac{6}{2s-3} - \dfrac{3+4s}{9s^2-16} + \dfrac{8-6s}{16s^2+9}\right\}$

Sol. $L^{-1}\left\{\dfrac{6}{2s-3} - \dfrac{3+4s}{9s^2-16} + \dfrac{8-6s}{16s^2+9}\right\}$

$$= L^{-1}\left\{\frac{6}{2(s-3/2)} - \frac{3}{9s^2-16} - \frac{4s}{9s^2-16} + \frac{8}{16s^2+9} - \frac{6s}{16s^2+9}\right\}$$

$$= L^{-1}\left\{\frac{6}{2(s-3/2)} - \frac{3}{9[s^2-(4/3)^2]} - \frac{4s}{9[s^2-(4/3)^2]} + \frac{8}{16[s^2+(3/4)^2]} - \frac{6s}{16[s^2+(3/4)^2]}\right\}$$

$$= L^{-1}\left\{\frac{3}{s-3/2} - \frac{3}{9}\,\frac{3}{4}\,\frac{4/3}{s^2-(4/3)^2} - \frac{4}{9}\,\frac{s}{s^2-(4/3)^2} + \frac{8}{16}\,\frac{4}{3}\,\frac{3/4}{s^2+(3/4)^2} - \frac{6}{16}\,\frac{s}{s^2+(3/4)^2}\right\}$$

$$= 3\,L^{-1}\left\{\frac{1}{s-3/2}\right\} - \frac{1}{4}\,L^{-1}\left\{\frac{4/3}{s^2-(4/3)^2}\right\} - \frac{4}{9}\,L^{-1}\left\{\frac{s}{s^2-(4/3)^2}\right\} + \frac{2}{3}\,L^{-1}\left\{\frac{3/4}{s^2+(3/4)^2}\right\} - \frac{3}{8}\,L^{-1}\left\{\frac{s}{s^2+(3/4)^2}\right\}$$

$$= 3\,e^{3t/2} - \frac{1}{4}\sinh\frac{4t}{3} - \frac{4}{9}\cosh\frac{4}{3}t + \frac{2}{3}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$$

$$\left[\because\ L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at},\ L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at,\ L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at,\ L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,\ L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at\right]$$

2.5 First Translation or Shifting Property:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s-b)\} = e^{bt}\,f(t)$

Proof: By definition, we have $F(s) = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt \quad \Rightarrow \quad L^{-1}\{F(s)\} = f(t)$

$$\therefore \quad F(s-b) = \int_0^\infty f(t)\,e^{-(s-b)t}\,dt = \int_0^\infty \left\{e^{bt}f(t)\right\}e^{-st}\,dt = L\{e^{bt}\,f(t)\}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\{F(s-b)\} = e^{bt}\,f(t)$$

i. e., $L^{-1}\{F(s-b)\} = e^{bt}\,L^{-1}\{F(s)\}$

Examples:

1. Find $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

Sol. $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\} = L^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$

$$= 6\,L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + \frac{8}{4}\,L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

$$= 6\,e^{2t}\,L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 2\,e^{2t}\,L^{-1}\left\{\frac{4}{s^2+4^2}\right\} \qquad [\because\ L^{-1}\{F(s-b)\} = e^{bt}\,L^{-1}\{F(s)\}]$$

$$= 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,\ L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at\right]$$

$$= 2\,e^{2t}(3\cos 4t + \sin 4t)$$

2. Find $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

Sol. $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2}\,(s+3/2)^{1/2}}\right\} = \frac{1}{\sqrt{2}}\,L^{-1}\left\{\frac{1}{\sqrt{s-(-3/2)}}\right\}$

$$= \frac{1}{\sqrt{2}}\,e^{-3t/2}\,L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} \qquad [\because\ L^{-1}\{F(s-b)\} = e^{bt}\,L^{-1}\{F(s)\}]$$

$$= \frac{1}{\sqrt{2}}\,e^{-3t/2}\,\frac{1}{\sqrt{\pi}}\,L^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}}\right\}$$

$$= \frac{1}{\sqrt{2}}\,e^{-3t/2}\,\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{t}} \qquad \left[\because\ L^{-1}\left\{\sqrt{\frac{\pi}{s}}\right\} = \frac{1}{\sqrt{t}}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,e^{-3t/2}\,t^{-1/2}$$

OR

We know that $L\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty t^{-1/2}\,e^{-st}\,dt = \frac{1}{s^{-(1/2)+1}}\,\Gamma\left(\frac{-1}{2}+1\right) = \sqrt{\frac{\pi}{s}}$

$$\left[\because\ \int_0^\infty t^{-n}\,e^{-st}\,dt = \frac{\Gamma(n+1)}{s^{n+1}}\ \&\ \Gamma(1/2) = \sqrt{\pi}\right]$$

$\therefore \quad L\left\{\frac{e^{-at}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s+a}} \qquad [\because$ if $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)]$

$\Rightarrow \quad L^{-1}\sqrt{\frac{\pi}{s+a}} = \frac{e^{-at}}{\sqrt{t}}$

$\Rightarrow \quad L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} = \frac{e^{-at}}{\sqrt{\pi t}}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\,L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} = \frac{1}{\sqrt{2}}\,\frac{e^{-at}}{\sqrt{\pi t}}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\,L^{-1}\left\{\frac{1}{\sqrt{s+3/2}}\right\} = \frac{1}{\sqrt{2}}\,\frac{e^{-3t/2}}{\sqrt{\pi t}} \qquad$ [putting $a = 3/2$]

$$\Rightarrow \qquad L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = \frac{1}{\sqrt{2\pi}}\, e^{-3t/2}\, t^{-1/2}$$

3. Evaluate $L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

Sol. $$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\} = L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-2^2}\right\}$$

$$= 3\,L^{-1}\left\{\frac{(s-1)}{(s-1)^2-2^2}\right\} + \frac{10}{2}\,L^{-1}\left\{\frac{2}{(s-1)^2-2^2}\right\}$$

$$= 3\,e^t\,L^{-1}\left\{\frac{s}{s^2-2^2}\right\} + 5\,e^t\,L^{-1}\left\{\frac{2}{s^2-2^2}\right\} \qquad [\because\ L^{-1}\{F(s-b)\} = e^{bt}\,L^{-1}\{F(s)\}]$$

$$= 3\,e^t\cosh 2t + 5\,e^t\sinh 2t \qquad \left[\because\ L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at,\ L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at\right]$$

2.6 Second Translation or Shifting Property:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = g(t)$,

where, $g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \qquad \Rightarrow \qquad L^{-1}\{F(s)\} = f(t)$$

$$\therefore \qquad e^{-as}F(s) = e^{-as}\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, e^{-s(t+a)}\, dt$$

$$= \int_0^\infty f(t+a-a)\, e^{-s(t+a)}\, dt \qquad \text{Let } t+a = u \ \therefore \quad dt = du$$

$$= \int_a^\infty f(u-a)\, e^{-su}\, du$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt$$

$$= \int_0^a (0)\, e^{-st}\, dt + \int_a^\infty f(t-a)\, e^{-st}\, dt$$

$$= \int_0^\infty g(t)\, e^{-st}\, dt; \qquad g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$$

$$= L\{g(t)\}$$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1}\{e^{-as}F(s)\} = g(t) = \begin{cases} f(t-a) & ; \quad t > a \\ 0 & ; \quad t < a \end{cases}$$

Examples:

1. Find $L^{-1}\left\{\dfrac{e^{-5s}}{(s-2)^4}\right\}$

Sol. Let us first find $L^{-1}\left\{\dfrac{1}{(s-2)^4}\right\}$:

$$L^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\, L^{-1}\left\{\frac{1}{s^4}\right\} \qquad [\because\ L^{-1}\{F(s-b)\} = e^{bt}\, L^{-1}\{F(s)\}]$$

$$= e^{2t}\,\frac{1}{\Gamma(4)}\, L^{-1}\left\{\frac{\Gamma(4)}{s^4}\right\}$$

$$= e^{2t}\,\frac{1}{3!}\, t^3 \qquad \left[\because\ L^{-1}\left\{\frac{\Gamma(n)}{s^n}\right\} = t^{n-1}\right]$$

$$= \frac{1}{6}\, e^{2t}\, t^3$$

By Using Second Translation Property, we get

$$L^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \begin{cases} \dfrac{1}{6} e^{2(t-5)}(t-5)^3 & ; \quad t > 5 \\ 0 & ; \quad t < 5 \end{cases}$$

2. Find $L^{-1}\left\{\dfrac{(s+1)\, e^{-\pi s}}{s^2+s+1}\right\}$

Sol. Let us first find $L^{-1}\left\{\dfrac{s+1}{s^2+s+1}\right\}$:

$$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{s+1/2+1/2}{(s+1/2)^2+3/4}\right\}$$

$$= L^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}\,\frac{2}{\sqrt{3}}\, L^{-1}\left\{\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\}$$

$$= e^{-t/2}\, L^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{\sqrt{3}}\, e^{-t/2}\, L^{-1}\left\{\frac{\sqrt{3}/2}{s^2+(\sqrt{3}/2)^2}\right\}$$

$$[\because\ L^{-1}\{F(s-b)\} = e^{bt}\, L^{-1}\{F(s)\}]$$

$$= e^{-t/2} \cos\frac{t\sqrt{3}}{2} + \frac{1}{\sqrt{3}} e^{-t/2} \sin\frac{t\sqrt{3}}{2}$$

$$= e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$

By Using Second Translation Property, we get

$$L^{-1}\left\{\frac{(s+1)\,e^{-\pi s}}{s^2+s+1}\right\} = \begin{cases} e^{-\frac{(t-\pi)}{2}}\left\{\cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-\pi)\right\}; & t > \pi \\ 0 \qquad ; & t < \pi \end{cases}$$

2.7 Inverse Laplace Transform of Derivatives:

If $L^{-1}\{F(s)\} = f(t)$ then

(i) $$L^{-1}\left\{\frac{d\,F(s)}{ds}\right\} = -\,t\,f(t)$$

(ii) $$L^{-1}\left\{\frac{d^2F(s)}{ds^2}\right\} = (-1)^2\,t^2 f(t)$$

Proof: (i) By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$\therefore \quad \frac{d\,F(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty f(t)\,(-t)\,e^{-st}\,dt$$

$$= -\int_0^\infty [t\,f(t)]\,e^{-st}\,dt \qquad (2.1)$$

$$= -\,L\,\{t\,f(t)\}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\left\{\frac{d\,F(s)}{ds}\right\} = -\,t\,f(t)$$

i. e., $$L^{-1}\left\{\frac{d\,F(s)}{ds}\right\} = -\,t\,L^{-1}\{F(s)\}$$

i. e., $$L^{-1}\{F(s)\} = \frac{-1}{t}\,L^{-1}\left\{\frac{d\,F(s)}{ds}\right\}$$

(ii) Differentiating both sides of eq. (2.1) w. r. t. s, we get

$$\frac{d^2F(s)}{ds^2} = -\frac{d}{ds}\int_0^\infty [t\,f(t)]\,e^{-st}\,dt = -\int_0^\infty (-t)\,[t\,f(t)]\,e^{-st}\,dt$$

$$= (-1)^2\int_0^\infty [t^2 f(t)]\,e^{-st}\,dt$$

$$= (-1)^2\,L\,\{t^2\,f(t)\}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\left\{\frac{d^2F(s)}{ds^2}\right\} = (-1)^2\, t^2 f(t)$$

Generalizing, $L^{-1}\left\{\frac{d^nF(s)}{ds^n}\right\} = (-1)^n\, t^n\, f(t)$

Examples:

1. Find the inverse Laplace transform of $\cot^{-1}\left(\frac{s+3}{2}\right)$

Sol. $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s+3}{2}\right)\right] \qquad \left[\because L^{-1}\{F(s)\} = \frac{-1}{t} L^{-1}\left\{\frac{d\,F(s)}{ds}\right\}\right]$$

$$= \frac{-1}{t} L^{-1}\left\{\frac{-(1/2)}{1+(s+3)^2/2^2}\right\}$$

$$= \frac{1}{2t} L^{-1}\left\{\frac{2^2}{(s+3)^2+2^2}\right\}$$

$$= \frac{1}{2t} 2\, L^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$$

$$= \frac{1}{t} e^{-3t} L^{-1}\left\{\frac{2}{s^2+2^2}\right\} \qquad [\because L^{-1}\{F(s-b)\} = e^{bt} L^{-1}\{F(s)\}]$$

$$= \frac{e^{-3t}}{t}\sin 2t \qquad \left[\because L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin\, at\right]$$

2. Find $L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right]$

Sol. $L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right] = L^{-1}\left[\log\left(\frac{s^2+1}{s^2}\right)\right] = L^{-1}\left[\log(s^2+1) - 2\log s\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - 2\log s\}\right] \qquad \left[\because L^{-1}\{F(s)\} = \frac{-1}{t} L^{-1}\left\{\frac{d\,F(s)}{ds}\right\}\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right]$$

$$= \frac{-2}{t} L^{-1}\left[\frac{s}{s^2+1^2} - \frac{1}{s}\right]$$

$$= \frac{-2}{t}(\cos t - 1) \qquad \left[\because L^{-1}\left\{\frac{1}{s}\right\} = 1,\ L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t\right]$$

$$= \frac{2}{t}(1 - \cos t)$$

2.8 Inverse Laplace Transform of Integrals:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t}$

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\therefore \quad \int_s^\infty F(u)\,du = \int_s^\infty \left(\int_0^\infty f(t)\, e^{-ut}\, dt\right) du$$

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-ut}\, du\right) dt$$

$$= \int_0^\infty f(t)\left(\frac{e^{-ut}}{-t}\right)_s^\infty dt$$

$$= \int_0^\infty \frac{-f(t)}{t}[0 - e^{-st}]dt$$

$$= \int_0^\infty \frac{f(t)}{t}\, e^{-st}\, dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t}$$

Examples:

1. If $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$, find $L^{-1}\left\{\int_s^\infty \frac{du}{u(u+1)}\right\}$.

Sol. Given $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$

$$\therefore \quad L^{-1}\left\{\int_s^\infty \frac{du}{u(u+1)}\right\} = \frac{1}{t}(1 - e^{-t}) \qquad \left[\because \text{ if } L^{-1}\{F(s)\} = f(t),\ L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t}\right]$$

2. If $L^{-1}\left\{\frac{2a^2 s}{s^4 + 4a^4}\right\} = \sinh at \sin at$, find $L^{-1}\left\{\int_s^\infty \frac{u\,du}{u^4 + 4a^4}\right\}$.

Sol. Given $L^{-1}\left\{\frac{2a^2 s}{s^4+4a^4}\right\} = \sinh at \sin at$

$\therefore \quad L^{-1}\left\{\int_s^\infty \frac{2a^2 u\,du}{u^4+4a^4}\right\} = \frac{1}{t}\sinh at \sin at$

$$\left[\because \text{ if } L^{-1}\{F(s)\} = f(t),\ L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t}\right]$$

$\Rightarrow \quad L^{-1}\left\{\int_s^\infty \frac{u\,du}{u^4+4a^4}\right\} = \frac{1}{2a^2 t}\sinh at \sin at$

2.9 Multiplication by the powers of s:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{s\,F(s) - f(0)\} = f'(t)$

If $f(0) = 0$, then $L^{-1}\{s\,F(s)\} = f'(t)$

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$\therefore \quad L\{f'(t)\} = \int_0^\infty f'(t)\,e^{-st}\,dt = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty -s\,e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\,L\{f(t)\}$$

$$= s\,F(s) - f(0)$$

Taking Inverse Laplace Transform on both sides, we get

$$L^{-1}\{s\,F(s) - f(0)\} = f'(t)$$

clearly, if $f(0) = 0$, then $L^{-1}\{s\,F(s)\} = f'(t)$

Examples:

1. Find $L^{-1}\left\{\frac{s}{s^2-6s+25}\right\}$.

Sol. Let us first find $L^{-1}\left\{\frac{1}{s^2-6s+25}\right\}$:

We have, $L^{-1}\left\{\frac{1}{s^2-6s+25}\right\} = L^{-1}\left\{\frac{1}{(s-3)^2+4^2}\right\} = \frac{1}{4}L^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$

$= \frac{1}{4}e^{3t} L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$ $\qquad [\because L^{-1}\{F(s-b)\} = e^{bt} L^{-1}\{F(s)\}]$

$= \frac{1}{4}e^{3t}\sin 4t = f(t)$ and $f(0) = 0$ $\qquad \left[\because L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at\right]$

$\therefore \quad L^{-1}\left\{s\,\frac{1}{s^2-6s+25}\right\} = f'(t) = \frac{d}{dt}\left(\frac{e^{3t}}{4}\sin 4t\right)$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{s^2-6s+25}\right\} = \frac{1}{4}\,(e^{3t}\,4\cos 4t + 3e^{3t}\sin 4t)$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{s^2-6s+25}\right\} = \frac{1}{4}\,e^{3t}\,(4\cos 4t + 3\sin 4t)$$

2. Find $L^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$.

Sol. Let us first find $L^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\}$:

We have, $L^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \dfrac{-1}{t}\,L^{-1}\left\{\dfrac{d}{ds}\left(\dfrac{1}{s^2+4}\right)\right\}$ $\quad\left[\because\ L^{-1}\{F(s)\} = \dfrac{-1}{t}L^{-1}\left\{\dfrac{d\,F(s)}{ds}\right\}\right]$

$$\Rightarrow \quad \frac{1}{2}\,L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{-1}{t}\,L^{-1}\left\{\frac{-2s}{(s^2+4)^2}\right\}$$

$$\Rightarrow \quad \frac{1}{2}\sin 2t = \frac{2}{t}\,L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} \qquad \left[\because\ L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at\right]$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t}{4}\sin 2t = f(t) \text{ and } f(0) = 0$$

$$\therefore \quad L^{-1}\left\{s\,\frac{s}{(s^2+4)^2}\right\} = f'(t) = \frac{d}{dt}\left(\frac{t}{4}\sin 2t\right)$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \frac{1}{4}\left(t\,2\cos 2t + \sin 2t\right)$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \frac{1}{2}\,t\cos 2t + \frac{1}{4}\sin 2t$$

2.10 Division by s:

If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{\dfrac{F(s)}{s}\right\} = \displaystyle\int_0^t f(u)\,du$

Proof: Let $\quad g(t) = \displaystyle\int_0^t f(u)\,du$

then $\quad g'(t) = f(t)$, with $g(0) = 0$

$\therefore \quad L\{g'(t)\} = s\,L\{g(t)\} - g(0)$

$\Rightarrow \quad L\{f(t)\} = s\,L\{g(t)\} - 0$

$\Rightarrow \quad F(s) = s\,L\{g(t)\}$

$\Rightarrow \quad L\{g(t)\} = \dfrac{F(s)}{s}$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = g(t) = \int_0^t f(u)\,du$$

Examples:

1. Evaluate $L^{-1}\left\{\dfrac{1}{s^2(s^2+a^2)}\right\}$.

Sol. $L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \dfrac{1}{a}\sin at = f(t)$

$\therefore \quad L^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\} = \displaystyle\int_0^t f(u)\,du = \int_0^t \frac{1}{a}\sin au\,du$

$$\left[\because \text{ if } L^{-1}\{F(s)\} = f(t), \text{ then } L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du\right]$$

$\Rightarrow \quad L^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\} = \left[\dfrac{-1}{a^2}\cos au\right]_0^t = \left[\dfrac{-1}{a^2}(\cos at - 1)\right]$

$$= \frac{1}{a^2}(1-\cos at) = g(t)$$

$\therefore \quad L^{-1}\left\{\dfrac{1}{s^2(s^2+a^2)}\right\} = \displaystyle\int_0^t g(u)\,du = \int_0^t \frac{1}{a^2}(1-\cos au)\,du$

$$\left[\because \text{ if } L^{-1}\{F(s)\} = g(t), \text{ then } L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t g(u)\,du\right]$$

$\Rightarrow \quad L^{-1}\left\{\dfrac{1}{s^2(s^2+a^2)}\right\} = \dfrac{1}{a^2}\left[u - \dfrac{\sin au}{a}\right]_0^t = \dfrac{1}{a^2}\left[t - \dfrac{\sin at}{a}\right]$

$$= \frac{1}{a^3}\left[at - \sin at\right]$$

2. Given $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t}{2}\sin t$, find $L^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$.

Sol. Given $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t}{2}\sin t = f(t)$

$\therefore \quad L^{-1}\left\{\dfrac{1}{s}\dfrac{s}{(s^2+1)^2}\right\} = \displaystyle\int_0^t f(u)\,du = \int_0^t \frac{u}{2}\sin u\,du$

$$\left[\because \text{ if } L^{-1}\{F(s)\} = f(t), \text{ then } L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du\right]$$

$\Rightarrow \quad L^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\} = \dfrac{1}{2}\left[u(-\cos u) - 1(-\sin u)\right]_0^t$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\left[(-1)(t\cos t - 0) + (\sin t - 0)\right]$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\left[\sin t - t\cos t\right]$$

3. Find $L^{-1}\left[\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right]$

Sol. We have found that

$$L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)\right] = \frac{2}{t}(1-\cos t) = f(t)$$

$$\therefore \quad L^{-1}\left[\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right] = \int_0^t f(u)\,du = \int_0^t \frac{2(1-\cos u)}{u}\,du$$

2.11 Convolution theorem:

If $f(t)$ and $g(t)$ are two functions and if $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$,

then $L^{-1}\{F(s)\,G(s)\} = f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du = \int_0^t f(t-u)\,g(u)\,du$

where, $f * g$ is called the convolution of f and g

Proof: By definition, we have

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt \quad \text{and} \quad G(s) = L\{g(t)\} = \int_0^\infty g(t)\,e^{-st}\,dt$$

$$\therefore \quad F(s)\,G(s) = \left\{\int_0^\infty f(u)\,e^{-su}\,du\right\}\left\{\int_0^\infty g(v)\,e^{-sv}\,dv\right\}$$

$$= \int_0^\infty\int_0^\infty e^{-s(u+v)}\,f(u)\,g(v)\,du\,dv \qquad \text{Let} \quad u+v=t \quad \therefore\ dv = dt$$

$$= \int_0^\infty dt \int_0^t du\; e^{-st}\,f(u)\,g(t-u)$$

$$= \int_0^\infty dt\; e^{-st}\left\{\int_0^t du\,f(u)\,g(t-u)\right\}$$

$$= L\left\{\int_0^t f(u)\,g(t-u)\,du\right\}$$

$$= L\left\{f(t) * g(t)\right\}$$

Taking Inverse Laplace Transform of both sides, we get

$$L^{-1}\{F(s)\,G(s)\} = f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Properties of Convolution:

1. Commutativity $f * g = g * f$

Proof: LHS = $f(t) * g(t) = \int_0^t f(u) g(t-u)\, du$

$$= \int_t^0 f(t-y) g(y)(-dy) \qquad [\text{Putting } t-u = y \ \therefore\ du = -dy]$$

$$= \int_0^t f(t-y)\, g(y)\, dy$$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du = g(t) * f(t) = \text{RHS}$$

Hence, the convolution of f and g obeys the commutative law.

2. Associativity $f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$

Proof: Let $f(t) * [g(t) * h(t)] = f(t) * m(t)$, where $m(t) = g(t) * h(t)$

By Definition, $m(t) = g(t) * h(t) = \int_0^t g(u) h(t-u)\, du = \int_0^t h(u) g(t-u)\, du$

$$\therefore \qquad f(t) * m(t) = \int_0^t f(y)\, m(t-y)\, dy$$

$$\Rightarrow \qquad f(t) * [g(t) * h(t)] = \int_0^t f(y)\, dy \left[\int_0^{t-y} h(u)\, g(t-y-u)\, du\right]$$

$$= \int_0^t h(u)\, du \int_0^{t-u} f(y)\, g(t-y-u)\, dy$$

[changing the order of integration]

$$= h(t) * [f(t) * g(t)]$$

$$= [f(t) * g(t)] * h(t) \qquad [\text{using commutativity property}]$$

3. Distributive with respect to addition $f * (g + h) = f * g + f * h$

Proof: $f * (g + h) = \int_0^t f(u)\, [g(t-u) + h(t-u)]\, du$

$$= \int_0^t f(u) g(t-u)\, du + \int_0^t f(u) h(t-u)\, du$$

$$= f * g + f * h$$

Examples:

1. Use Convolution Theorem to find $L^{-1}\left\{\dfrac{s}{(s^2 + a^2)^2}\right\}$.

Sol. Let $F(s) = \dfrac{1}{(s^2 + a^2)}$, $G(s) = \dfrac{s}{(s^2 + a^2)}$

$\therefore$ $f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{s^2 + a^2}\right\} = \dfrac{1}{a}\sin at$

and $g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\dfrac{s}{s^2 + a^2}\right\} = \cos at$

Now, using convolution theorem, we get

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{(s^2 + a^2)}\cdot\frac{s}{(s^2 + a^2)}\right\} = \int_0^t \frac{1}{a}\sin au\,[\cos a(t-u)]\,du$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\int_0^t 2\sin au\,\cos(at - au)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin(au + at - au) + \sin(au - at + au)]du$$

$$= \frac{1}{2a}\int_0^t [\sin(at) + \sin(2au - at)]du$$

$$= \frac{1}{2a}\left[\sin(at)\int_0^t du + \int_0^t \sin(2au - at)du\right]$$

$$= \frac{1}{2a}\left[t\sin(at) - \frac{\cos(2au - at)}{2a}\bigg|_0^t\right]$$

$$= \frac{t\sin at}{2a} - \frac{(\cos at - \cos at)}{4a^2}$$

$$= \frac{t\sin at}{2a}$$

2. Use Convolution Theorem to find $L^{-1}\left\{\dfrac{1}{s^2\,(s+1)^2}\right\}$.

Sol. Let $F(s) = \dfrac{1}{s^2}$, $G(s) = \dfrac{1}{(s+1)^2}$

$\therefore$ $f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{s^2}\right\} = t$

and $g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\} = t\,e^{-t}$

Now, using convolution theorem, we get

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right\} = \int_0^t u\,e^{-u}(t-u)\,du$$

$$= t\int_0^t u\,e^{-u}\,du - \int_0^t u^2 e^{-u}\,du$$

$$= t\,[-u\,e^{-u} - e^{-u}]_0^t - [-u^2 e^{-u} - 2u\,e^{-u} - 2e^{-u}]_0^t$$

$$= t\,[-t\,e^{-t} - (e^{-t}-1)] - [-t^2 e^{-t} - 2t\,e^{-t} - 2(e^{-t}-1)]$$

$$= t\,e^{-t} + 2e^{-t} + t - 2$$

Partial Fractions:

Sometimes the partial fraction method is very useful in finding the Inverse Laplace Transform.

Examples:

1. Evaluate $L^{-1}\left\{\dfrac{s}{s^4+s^2+1}\right\}$.

Sol. Now, $\dfrac{s}{s^4+s^2+1} = \dfrac{s}{(s^2+1)^2 - s^2} = \dfrac{s}{(s^2+1+s)(s^2+1-s)}$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] \qquad \text{[Resolving into partial fractions]}$$

$$\therefore \quad L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{1}{2}\,L^{-1}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$

$$= \frac{1}{2}\,L^{-1}\left[\frac{1}{(s-1/2)^2+3/4} - \frac{1}{(s+1/2)^2+3/4}\right]$$

$$= \frac{1}{2}\,e^{t/2}\,L^{-1}\left\{\frac{1}{s^2+(\sqrt{3}/2)^2}\right\} - \frac{1}{2}\,e^{-t/2}\,L^{-1}\left\{\frac{1}{s^2+(\sqrt{3}/2)^2}\right\}$$

$$= \frac{1}{2}\,e^{t/2}\,\frac{2}{\sqrt{3}}\,L^{-1}\left\{\frac{\sqrt{3}/2}{s^2+(\sqrt{3}/2)^2}\right\} - \frac{1}{2}\,e^{-t/2}\,\frac{2}{\sqrt{3}}\,L^{-1}\left\{\frac{\sqrt{3}/2}{s^2+(\sqrt{3}/2)^2}\right\}$$

$$= e^{t/2}\,\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right) - e^{-t/2}\,\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

$$= \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)(e^{t/2} - e^{-t/2})$$

$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)\sinh\left(\frac{t}{2}\right)$$

2. Find $L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$

Sol. $$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = L^{-1}\left\{\frac{A}{s-1}+\frac{Bs+C}{s^2+1}\right\} = L^{-1}\left\{\frac{A}{s-1}+\frac{Bs}{s^2+1}+\frac{C}{s^2+1}\right\}$$

$$= A\,L^{-1}\left\{\frac{1}{s-1}\right\} + B\,L^{-1}\left\{\frac{s}{s^2+1}\right\} + C\,L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= A\,e^t + B\cos t + C\sin t \qquad (2.2)$$

Now to find the constants A, B and C:

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$\Rightarrow \quad 3s+1 = A(s^2+1)+(Bs+C)(s-1)$

Put $s=1$, we get $4=2A \quad \Rightarrow \quad A=2$

Comparing the coefficients of s^2 on both sides, we get

$0 = A+B \quad \Rightarrow \quad B=-A=-2$

Put $s=0$, we get $1 = A - C \quad \Rightarrow \quad C = A-1 = 2-1 = 1$

Put $A=2$, $B=-2$ and $C=1$ in eq. (2.2), we get

$$\therefore \quad L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = 2e^t - 2\cos t + \sin t$$

Note: $h(t) = \int_0^t g(u)\,du \quad \Leftrightarrow \quad \dfrac{\partial}{\partial t}h(t) = g(t)$, provided $h(0)=0$.

Eg. $erf(t) = \dfrac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du \quad \Leftrightarrow \quad \dfrac{\partial}{\partial t}erf(t) = \dfrac{2}{\sqrt{\pi}}e^{-t^2}$

where $erf(t)$ is known as error function.