Ministry of Higher Education and Scientific Research
Almustaqbal University, College of Engineering
And Engineering Technologies
Department of Electrical Engineering

Three Week :
Step Response of Circuits

Course Name : Electrical Circuits Analysis
Stage : Second
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1) Step Response of an *RC* Circuit

When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response***.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

Consider the RC circuit in Fig.5.1 (a) which can be replaced by the circuit in Fig.5.1 (b), where V_s is a constant dc voltage source. We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response.

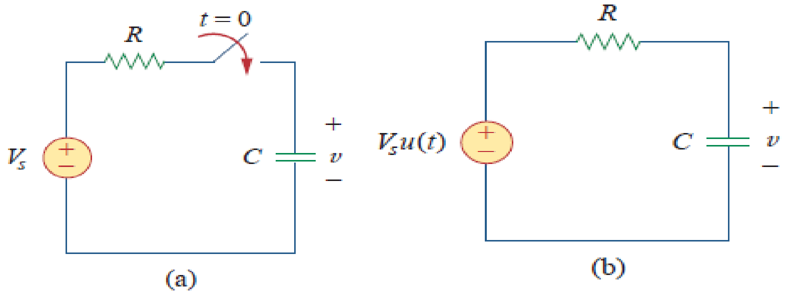


Fig.5.1 An RC circuit with voltage step input.

Since the voltage of a capacitor cannot change instantaneously,

$$v(0^-) = v(0^+) = V_0 \tag{5.1}$$

where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage immediately after switching. Applying KCL, we have

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t) \tag{5.2}$$

where v is the voltage across the capacitor. For $t > 0$, Eq. (5.2) becomes

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \tag{5.3}$$

Rearranging terms gives

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\therefore \quad \frac{dv}{v - V_s} = -\frac{dt}{RC} \tag{5.4}$$

Integrating both sides and introducing the initial conditions,

$$ln\,(v - V_s)|_{V_0}^{v(t)} = -\frac{t}{RC}|_0^t$$

$$ln\,(v(t) - V_s) - ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$ln\,\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC} \tag{5.5}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$\therefore v(t) = V_s + (V_0 - V_s)e^{-\frac{t}{\tau}}, \quad t > 0 \tag{5.6}$$

Thus,

$$\boxed{v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}} \tag{5.7}$$

This is known as the **complete response** (or **total response**) of the RC circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term "complete" will become evident a little later. Assuming that $V_s > V_0$, a plot of $v(t)$ is shown in Fig.5.2.

If we assume that the capacitor is uncharged initially, we set $V_0 = 0$ in Eq. (5.7) so that

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases} \tag{5.8}$$

which can be written alternatively as

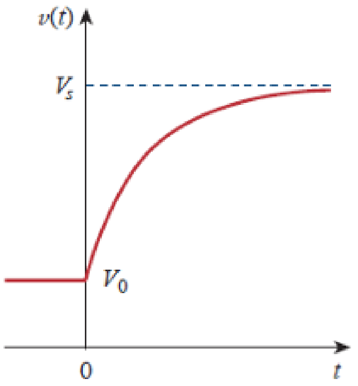
$$v(t) = V_s(1 - e^{-t/\tau})u(t) \tag{5.9}$$

This is the complete step response of the RC circuit when the capacitor is initially uncharged. The current through the capacitor is obtained from Eq. (5.8) using $i(t) = \frac{Cdv}{dt}$ —
We get

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-\frac{t}{\tau}}, \qquad \tau = RC, t > 0$$

$$\therefore\ i(t) = \frac{V_s}{R}e^{-t/\tau}u(t) \tag{5.10}$$

Fig.5.3 shows the plots of capacitor voltage $v(t)$ and capacitor current $i(t)$.



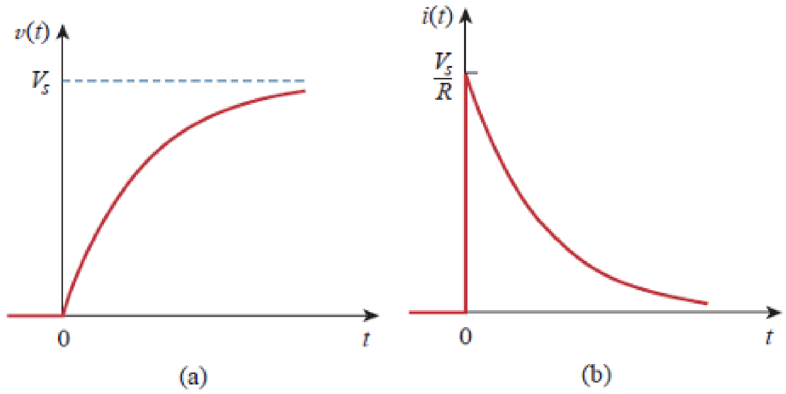


Fig.5.2 Response of an RC circuit with initially charged capacitor.

Fig.5.3 Step response of an RC circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an RC or RL circuit. Let us reexamine Eq. (5.6), which is more general than Eq. (5.9). It is evident that $v(t)$ has two components.

Classically there are two ways of decomposing this into two components. The first is to break it into a "***natural response and a forced response***" and the second is to break it into a "***transient response and a steady-state response***." Starting with the natural response and forced

(a) response, we write the total or complete response as

$$\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$$

or

$$v = v_n - v_f \tag{5.11}$$

where $v_n = V_o e^{-t/\tau}$ and $v_f = V_s(1 - e^{-t/\tau})$

We are familiar with the natural response v_n of the circuit, as discussed response, (b) current response. in Section 2. v_f is known as the ***forced response*** because it is produced by the circuit when an external "***force***" (a voltage source in this case) is applied. It represents what the circuit is forced to do by the input excitation. The natural response eventually dies out along with the transient component of the forced response, leaving only the steady-state component of the forced response.

Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

or

$$v = v_t - v_{ss} \tag{5.12}$$

where $v_t = (V_o - V_s)e^{-t/\tau}$ and $v_{ss} = V_s$

The ***transient response*** v_t is temporary; it is the portion of the complete response that decays to zero as time approaches infinity. Thus,

The transient response is the circuit's temporary response that will die out with time.

The ***steady-state response*** v_{ss} is the portion of the complete response that remains after the transient reponse has died out. Thus,

The steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses. Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response.

Whichever way we look at it, the complete response in Eq. (5.6) may be written as

$$v(t) = v(\infty) + [v(O) - v(\infty)]e^{-t/\tau} \tag{5.13}$$

where $v(O)$ is the initial voltage at $t = 0^+$ and $v(\infty)$ is the final or steady - state value.

To find the step response of an RC circuit requires three things:

1. The initial capacitor voltage $v(O)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

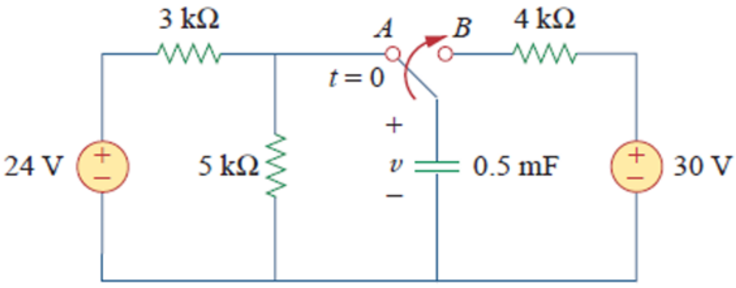
We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (5.13). This technique equally applies to RL circuits, as we shall see in the next section.

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that Eq. (5.13) becomes

$$v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau} \quad (5.14)$$

Example 10: The switch in Fig.1 has been in position A for along time. At $t = 0$, the switch moves to B. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1s$ and $4\ s$.

Solution: For $t < 0$, the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the $5 - k\Omega$ resistor. Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as



$$v(0^-) = \frac{5}{5+3}(24) = 15V$$

Fig.1

Using the fact that the capacitor voltage cannot change instantaneously,

$$v(O) = v(0^-) = v(0^+) = 15V$$

For $t > 0$, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4k\Omega$, and the time constant is

$$\tau = R_{Th}C = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2s$$

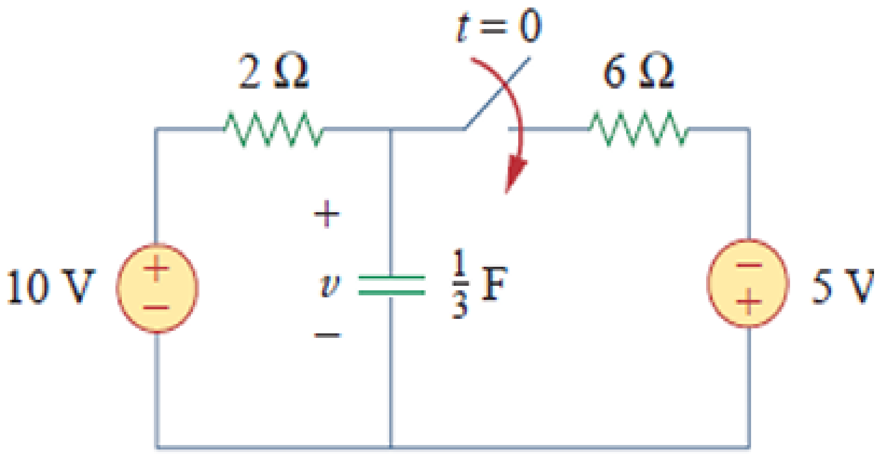
Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} = 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t})V$$

At $t = 1, \Rightarrow\ v(1) = 30 - 15e^{-0.5} = 20.9V$

At $t = 4, \Rightarrow\ v(4) = 30 - 15e^{-2} = 27.97V$

H.W. 10: Find $v(t)$ for $t > 0$ in the circuit of Fig.1. Assume the switch has been open for a long time and is closed at $t = 0$. Calculate $v(t)$ at $t = 0.5$



Answer: $(6.25 + 3.75e^{-2t})V$ **for all** $t > 0, 7.63V$.

Example 11: In Fig.1, the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.

Solution: The resistor current i can be discontinuous at $t = 0$, while the capacitor voltage v cannot. Hence, it is always better to find v and then obtain i from v.

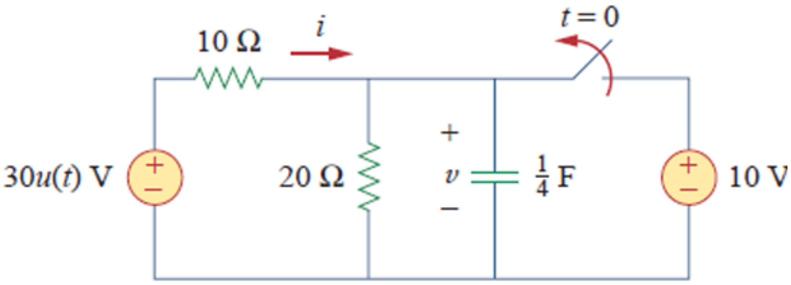


Fig.1

By definition of the unit step function,

$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$, the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit and should be regarded as contributing nothing to v. Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit. Hence, the circuit becomes that shown in Fig.2 (a)

for $t < 0$. From this circuit we obtain

$$v = 10V, \qquad i = -\frac{v}{10} = -1A$$

Since the capacitor voltage cannot change instantaneously,

$$v(O) = v(0^-) = 10V$$

For $t > 0$, the switch is opened and the 10-V voltage source is disconnected from the circuit. The $30u(t)$ voltage source is now operative, so the circuit becomes that shown in Fig.2 (b). After a long time, the circuit reaches steady state and the capacitor acts like an open circuit (b) again. We obtain $v(\infty)$ by using voltage division, writing

$$v(\infty) = \frac{20}{20+10}(30) = 20V$$

The Thevenin resistance at the capacitor terminals is

$$R_{Th} = 10 \| 20 = \frac{10 \times 20}{30} = \frac{20}{3}\Omega$$

and the time constant is

$$\tau = R_{Th}C = \frac{20}{3} \cdot \frac{1}{4} = \frac{5}{3}s$$

Thus,

$$v(t) = v(\infty) + [v(O) - v(\infty)]e^{-t/\tau}$$

$$= 20 + (10 - 20)e^{-\left(\frac{3}{5}\right)t}$$

$$= (20 - 10e^{-0.6t})V$$

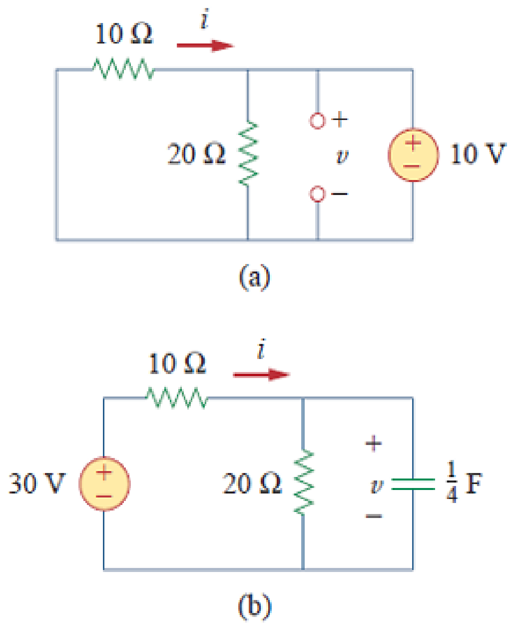


Fig.2 (a) for t<0, for t>0.

To obtain i, we notice from Fig.2 (b) that i is the sum of the currents through the $20-\Omega$ resistor and the capacitor; tha

$$i = \frac{v}{20} + C\frac{dv}{dt} = 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})A$$

Notice from Fig.2 (b) that $v + 10i = 30$ is satisfied, as expected. Hence,

$$v = \begin{cases} 10V, & t < 0 \\ (20 - 10e^{-0.6t})V, & t \geq 0 \end{cases} \quad i = \begin{cases} \text{-} 1A, & t < 0 \\ (1 + e^{-0.6t})A, & t > 0 \end{cases}$$

H.W. 11: The switch in Fig.1 is closed at $t = 0$. Find $i(t)$ and $u(t)$ for all time. Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.

Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ -2(1 + e^{-15t})A, & t > 0' \end{cases}$$

$$v = \begin{cases} 20V, & t < 0 \\ 10(1 + e^{-15t})V, & t > 0 \end{cases}$$

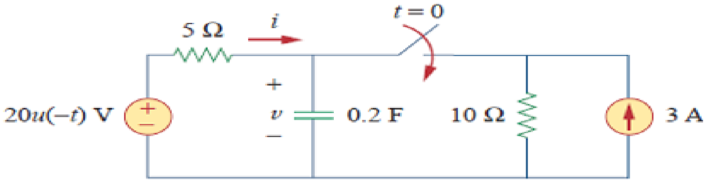


Fig.1

2) Step Response of an *RL* Circuit

Consider the *RL* circuit in Fig.6.1 (a), which may be replaced by the circuit in Fig.6.1 (b). Rather than apply Kirchhoff's laws, we will use the simple technique in Eqs. (5.11) through (5.14). Let the response be the sum of the transient response and the steady-state response,

$$i = i_t + i_{ss} \tag{6.1}$$

We know that the transient response is always a decaying exponential, that is,

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R} \tag{6.2}$$

where A is a constant to be determined.

The steady-state response is the value of the current a long time after the switch in Fig.6.1 (a) is closed. We know that the transient response essentially dies out after five time constants. At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R. Thus, the steady-state response is

$$i_{ss} = \frac{V_s}{R} \tag{6.3}$$

Substituting Eqs. (6.2) and (6.3) into Eq. (6.1) gives

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \tag{6.4}$$

We now determine the constant A ffom the initial value of i. Let I_0 be the initial current through the inductor, which may come from a source other than V_s. Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0 \tag{6.5}$$

Thus, at $t = 0$, Eq. (6.4) becomes

$$I_0 = A + \frac{V_s}{R}$$

From this, we obtain A as

$$A = I_0 - \frac{V_s}{R}$$

Substituting for A in Eq. (6.4), we get

$$i(t) = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-t/}\tau \qquad (6.6)$$

This is the complete response of the RL circuit. It is illustrated in Fig.6.2. The response in Eq. (6.6) may be written as

$$i(t) = i(\infty) + [i(O) - i(\infty)]e^{-t/\tau} \qquad (6.7)$$

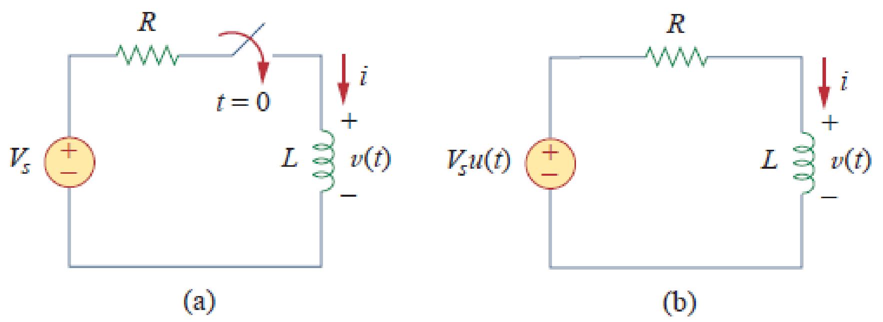


Fig.6.1 An RL circuit with a step input voltage.

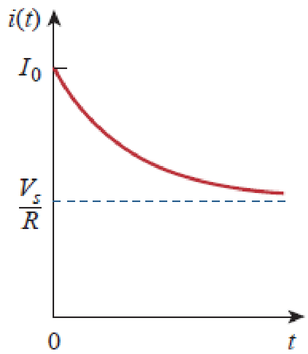


Fig.6.2 Total response of the RL circuit with initial inductor current I_0.

where $i(O)$ and $i(\infty)$ are the initial and final values of i, respectively. Thus, to find the step response of an RL circuit requires three things:

1. The initial inductor current $i(O)$ at $t = 0$.

2. The final inductor current $i(\infty)$.

3. The time constant τ.

We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these items are determined, we obtain the response using Eq. (6.7). Keep in mind that this technique applies only for step responses.

Again, if the switching takes place at time $t = t_0$ instead of $t = 0$, Eq. (6.7) becomes

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_o)/\tau} \qquad (6.8)$$

If $I_0 = 0$, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad (6.9a)$$

or

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t) \tag{6.9b}$$

This is the step response of the *RL* circuit with no initial inductor current. The voltage across the inductor is obtained from Eq. (6.9) using $v = Ldi/dt$. We get

$$v(t) = L\frac{di}{dt} = V_s \frac{L}{\tau R} e^{-t/\tau}, \tau = \frac{L}{R}, t > 0$$

or

$$v(t) = V_s e^{-t/\tau} u(t) \tag{6.10}$$

Fig.6.3 shows the step responses in Eqs. (6.9) and (6.10).

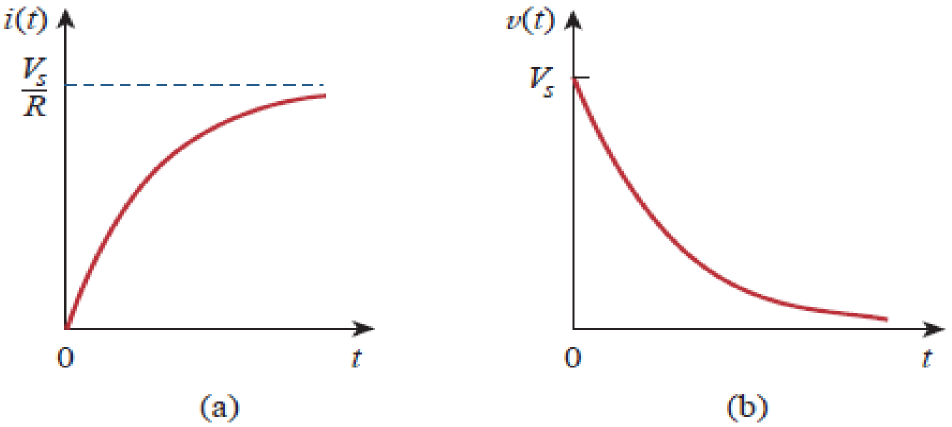


Fig.6.3 Step responses of an *RL* circuit with no initial inductor current: (a) current response, (b) voltage response.

Example 12: Find $i(t)$ in the circuit of Fig.1 for $t > 0$. Assume that the switch has been closed for a long time.

Solution: When $t < 0$, the $3 - \Omega$ resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor at $t = 0^-$(*i. e.*, just before $t = 0$) is

$$i(0^-) = \frac{10}{2} = 5A$$

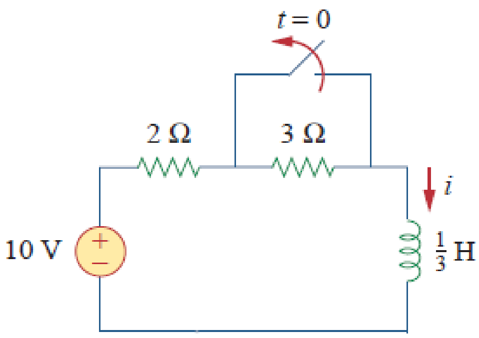


Fig.1

Since the inductor current cannot change instantaneously,

$i(O) = i(0^+) = i(0^-) = 5A$

When $t > 0$, the switch is open. The $2 - \Omega$ and $3 - \Omega$ resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2A$$

The Thevenin resistance across the inductor terminals is

$R_{Th} = 2 + 3 = 5\Omega$

For the time constant,

$$\tau = \frac{L}{R_{Th}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}s$$

Thus,

$i(t) = i(\infty) + [i(O) - i(\infty)]e^{-t/\tau}$

$= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t}A, t > 0$

Check. In Fig.1, for $t > 0$, KVL must be satisfied; that is,

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} = [10 + 15e^{-15t}] + [\frac{1}{3}(3)(-15)e^{-15t}] = 10$$

This confirms the result.

H.W. 12: The switch in Fig.1 has been closed for a long time. It opens at $t = 0$. Find $i(t)$ for $t > 0$.

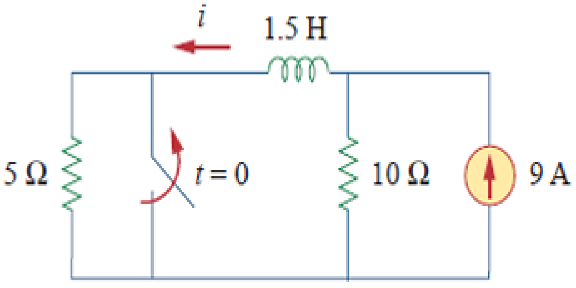


Fig.1

Answer: $(6 + 3e^{-10t})$ **A for all** $t > 0$.

Example 13: At $t = 0$, switch 1 in Fig.1is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2s$ and $t = 5s$.

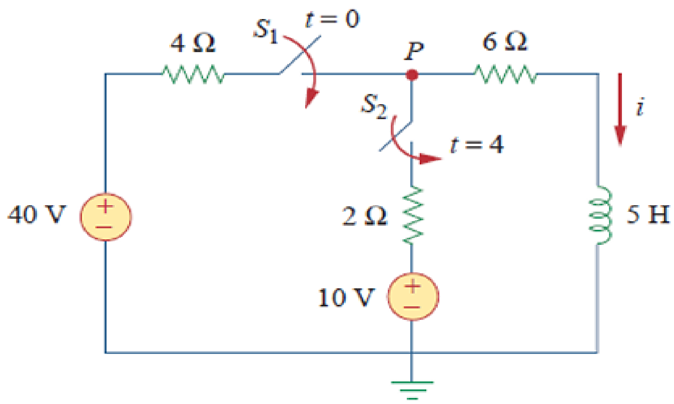


Fig.1

Solution: We need to consider the three time intervals $t \leq 0, 0 \leq t \leq 4$, and $t \geq 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$.
Since the inductor current cannot change instantly,

$$i(0^-) = i(O) = i(0^+) = 0$$

For $0 \leq t \leq 4$, S_1 is closed so that the $4-\Omega$ and $6-\Omega$ resistors are in series. (Remember, *at* this time, S_2 *is* still open.) Hence, assuming for now that
S_1 is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4A, \quad R_{Th} = 4 + 6 = 10\Omega$$

$$\tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{2}s$$

Thus,

$$i(t) = i(\infty) + [i(O) - i(\infty)]e^{-t/\tau} = 4 + (0-4)e^{-2t} = 4(1-e^{-2t})A, 0 \leq t \leq 4$$

For $t \geq 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1-e^{-8}) = 4A$$

To find $i(\infty)$, let v be the voltage at node P in Fig.1. Using KCL,

$$\frac{40-v}{4} + \frac{10-v}{2} = \frac{U}{6} \Rightarrow v = \frac{180}{11}V$$

$$i(\infty) = \frac{v}{6} = \frac{30}{11} = 2.727A$$

The Thevenin resistance at the inductor terminals is

$$R_{Th} = 4\|2 + 6 = \frac{4 \times 2}{6} + 6 = \frac{22}{3}\Omega$$

And

$$\tau = \frac{L}{R_{Th}} = \frac{5}{\frac{22}{3}} = \frac{15}{22}s$$

Hence,

$$i(t) = i(\infty) + [i(4) - i(\infty)]e^{-(t-4)/\tau}, t \geq 4$$

We need $(t - 4)$ in the exponential because of the time delay. Thus,

$$i(t) = 2.727 + (4 - 2.727)e^{-(t-4)/\tau}, \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-14667(t-4)}, t \geq 4$$

Putting all this together,

$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1 - e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 + 1.273e^{-14667(t-4)}, & t \geq 4 \end{cases}$$

At $t = 2$,

$$i(2) = 4(1 - e^{-4}) = 3.93A$$

At $t = 5$,

$$i(5) = 2.727 + 1.273e^{-14667} = 3.02A$$

H.W. 13: Switch S_1 in Fig.1 is closed at $t = 0$, and switch S_2 is closed at $t = 2s$. Calculate $i(t)$ for all t. Find $i(1)$ and $i(3)$.

Answer:

$$i(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-9t}), & 0 < t < 2 \\ 3.6 - 1.6e^{-5(t-2)}, & t > 2 \end{cases}$$

$i(1) = 1.9997A, i(3) = 3.589A.$

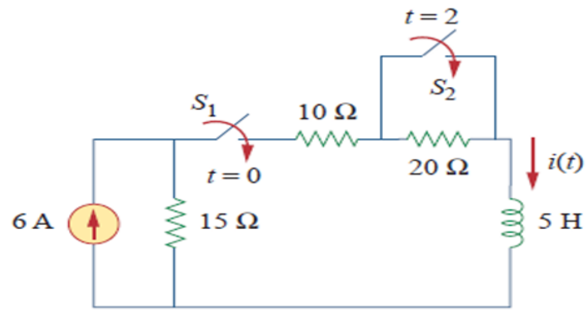


Fig.1

Thank you very much