Ministry of Higher Education and Scientific Research
Almustaqbal University, College of Engineering
And Engineering Technologies
Department of Electrical Engineering

One week :

Transient Applications

Course Name : Electrical Circuits Analysis
Stage : Second
Academic Year : 2024
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Transient Circuits

1. First-Order Circuits

1) The Source-Free Response

i. The Source-Free Of RC Circuit
ii. The Source-Free Of RL Circuit

2) Step Response

i. Step Response of RC Circuit
ii. Step Response of RL Circuit

2. Second-Order Circuits

1) The Source-Free Response RLC (Series & Parallel)

2) Step Response RLC (Series & Parallel)

1. First-Order Circuits

Introduction

The analysis of RC and RL circuits is carried out by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to RC and RL circuits produces differential equations, which are more difficult to solve than algebraic equations. The differential equations resulting from analyzing RC and RL circuits are of the first order. Hence, the circuits are collectively known as **first-order** circuits.

A first-order circuit is characterized by a first-order differential equation.

In addition to there being two types of first-order circuits (*RC* and *RL*), there are two waysto excite the circuits.

1) The first way is by initial conditions of the storage elements in the circuits. In these so-called ***source-free circuits***, we assume that energy is initially stored in the capacitive or inductive element. The energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source free circuits are by definition free of independent sources, they may have dependent sources.
2) The second way of exciting first-order circuits is by independent sources (dc and ac sources).

1.1 The Source-Free RC Circuit

A source-free *RC* circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

Consider a series combination of a resistor and an initially charged capacitor, as shown in Fig.2.1. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.) Now to determine the circuit response. Since the capacitor is initially charged, we can assume that at time $t = 0$, the initial voltage is source-free RC circuit

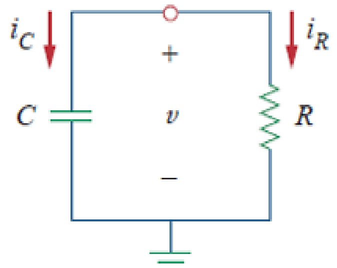


Fig.2.1.source-free RC circuit

$$v(0) = V_0 \qquad \ldots(2.1)$$

with the corresponding value of the energy stored as

$$w(0) = \frac{1}{2} C V_0^2 \qquad \ldots(2.2)$$

Applying KCL at the top node of the circuit in Fig. 2.1 yields

$$i_C + i_R = 0 \qquad \text{...(2.3)}$$

By definition, $i_C = Cdv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \qquad \text{...(2.4a)}$$

$$\therefore \frac{dv}{dt} + \frac{v}{RC} = 0 \qquad \text{...(2.4b)}$$

This is a **first-order differential equation**, since only the first derivative of u is involved. To solve it, we rearrange the terms as :

response of the RC circuit is an exponential decay of the initialvoltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the ***natural response*** of the circuit.

$$\frac{dv}{v} = -\frac{1}{RC}dt \qquad \text{...(2.5)}$$

Integrating both sides, we get

$$ln\ v = -\frac{t}{RC} + ln\ A$$

where $ln\ A$ is the integration constant. Thus,

$$ln\frac{v}{A} = -\frac{t}{RC} \qquad \text{...(2.6)}$$

Taking powers of e produces

$$v(t) = Ae^{-t/RC}$$

But from the initial conditions, $v(O) = A = V_0$. Hence,

$$v(t) = V_0e^{-t/RC} \qquad \text{...(2.7)}$$

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in Fig.2.2. Note that at $t = 0$, we have the correct initial condition as in Eq. (2.1). As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the ***time constant***, denoted by τ, the lowercase Greek letter tau.

The time constant of a circuit is the time required for the response to decay to afactor of 1/e or 36.8 percent of its initial value.

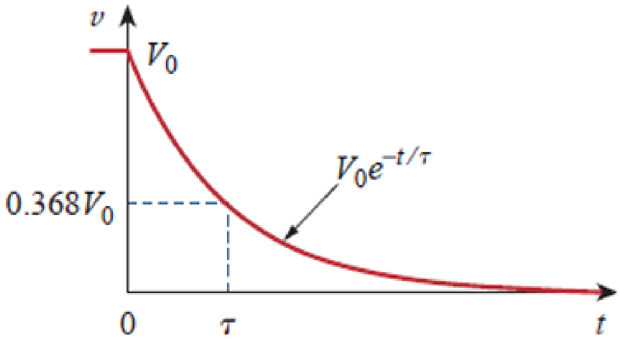


Fig.2.2. The voltage response of the RC circuit.

This implies that at $t = \tau$, Eq. (2.7) becomes

$$V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368V_0$$

or

$$\boxed{\tau = RC} \qquad \ldots(2.8)$$

In terms of the time constant, Eq. (2.7) can be written as

$$\boxed{v(t) = V_0 e^{-t/\tau}} \qquad \ldots(2.9)$$

It is evident from **Table 7.1** that the voltage $v(t)$ is less than 1 percent of V_0 after 5τ (five time constants).**Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five time constants.** In otherwords, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time. **Notice that for every time interval of τ, the voltage is reduced by 36.8 percent of its previous value,**

$v(t+\tau) = v(t)/e = 0.368v(t)$ **, regardless of the value of** t.

TABLE 2.1

Values of $v(t)/V_0 = e^{-t/\tau}$.

t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

A circuit with a small time constant gives a fast response in that it reaches the steady state (or finalstate) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constantgives a slow response because it takes longer to reach steady state (this is illustrated in (Fig. 2.3**). Atany rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.**

With the voltage $v(t)$ in Eq. (2.9), we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \quad \text{...(2.10)}$$

The power dissipated in the resistor is

$$p(t) = vi_R = \frac{V_0^2}{R} e^{-2t/\tau} \quad \text{...(2.11)}$$

The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt = -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),\ \tau = RC \quad .(2.12)$$

Notice that as $t \to \infty$, $W_0(\infty) \to \frac{1}{2} C V_0^2$ which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

With these two items, we obtain the response as the capacitor voltage

$$v_C(t) = v(t) = v(0) e^{-t/\tau}$$

other variables (capacitor current i_C, resistor voltage v_R, and resistor current i_R) can be determined. **In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor C and find $R = R_{Th}$ atits terminals.**

Example 1: In Fig. let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.

Solution:

We first need to make the circuit in Fig.1 conform with the standard RC circuit in Fig.2.1. We find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x.

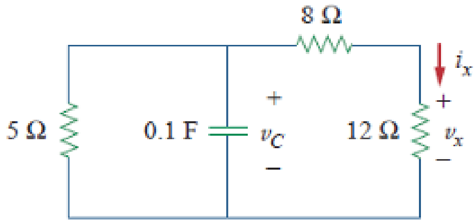


Fig.1

$$R_{eq} = (8+12) \parallel 5 \Rightarrow \therefore R_{eq} = \frac{20\times5}{20+5} = 4\Omega$$

∴ the equivalent circuit is as shown in Fig.2.

$\tau = R_{eq}C = 4(0.1) = 0.4s$

$v = v(0)e^{-t/\tau} = 15e^{-t/04}V.\ v_C = v = 15e^{-2.5t}V$

use voltage division to get v_x so,

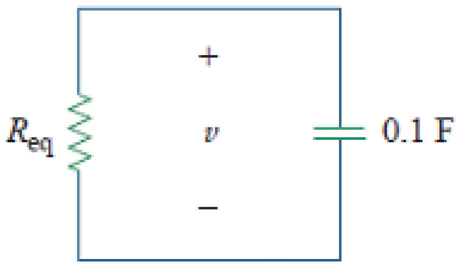


Fig.2

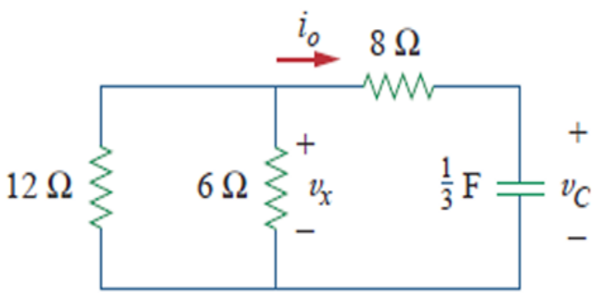
$$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}V$$

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}A$$

H.W.1: Let $v_C(0) = 45$ V. Determine v_C, v_x. and i_o for $f \geq 0$.

Answer:

$45e^{-0.25t}V.\ 15e^{-025t}V.\ -3.75e^{-0.25t}A.$



Example 2: The switch in the circuit in Fig.1 has been closed fora long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.

Solution:

For $t < 0$, the switch is closed; the capacitor is an open circuit to dc , as represented in Fig.2(a). Using voltage division

$$v_C(t) = \frac{9}{9+3}(20) = 15V.\ t < 0$$

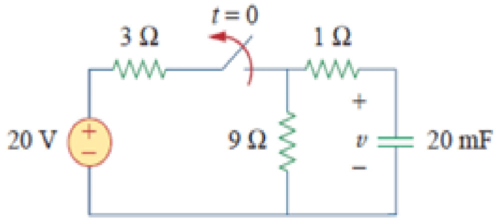


Fig. 1

Since the voltage across a capacitor cannot change instantaneously, the voltage across thecapacitor at $t = 0^-$ is the same at $t = 0$, or

$$v_C(0) = V_0 = 15V$$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig.2 (b). [Notice that the RC circuit in Fig.2 (b) is source free; the independent source in Fig.1 is needed to provide V_0 or the initial energy in the capacitor.]

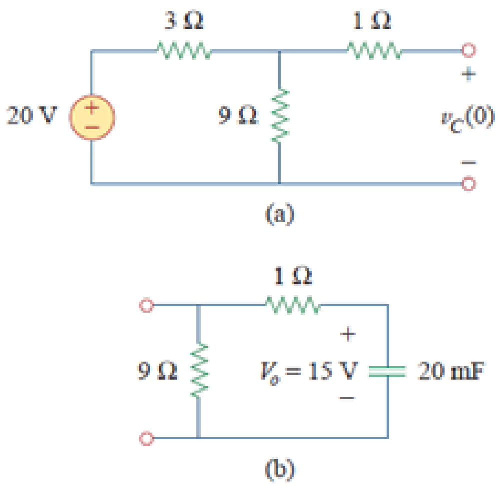


Fig.2

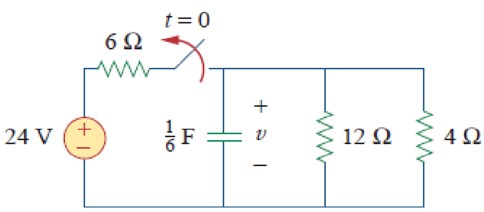
$$R_{eq} = 1+9 = 10\Omega\ ,\tau = R_{eq}C = 10\times20\times10^{-3} = 0\cdot2s$$

Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0\cdot2}V = 15e^{-5t}V$$

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2}\times 20\times10^{-3}\times 15^2 = 2.25J$$

H.W.2: If the switch in Fig. shown opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.



Answer: $8e^{-2t}V.\ 5.33J$

1.2 The Source-Free *RL* Circuit

Our goal is to determine the circuit response (current $i(t)$ through the inductor). We select the inductor current as the response in order to take advantage of the idea $\underline{i}$ that the inductor current cannot change instantaneously. At $t = 0$, we assume that the inductor has an initial current I_0, or with the corresponding energy stored in the inductor as

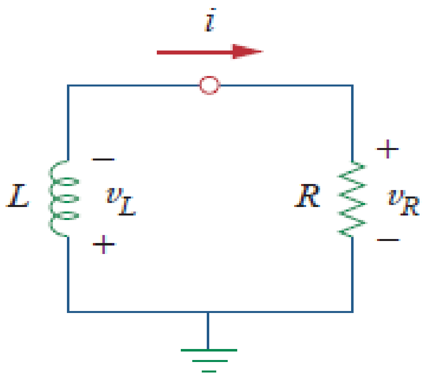


Fig.3.1 A source-free RL circuit

$$i(0) = I_o \tag{3.1}$$

$$w(0) = \frac{1}{2} L I_0^2 \tag{3.2}$$

Applying KVL around the loop in Fig.3.1,

$$v_L + v_R = 0 \tag{3.3}$$

But $v_L = L di/dt$ and $v_R = iR$. Thus,

$$L\frac{di}{dt} + Ri = 0 \quad \Rightarrow \quad \frac{di}{dt} + \frac{R}{L} i = 0 \tag{3.4}$$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt \quad \Rightarrow \quad \ln i \,\Big|_{I_o}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \Rightarrow \ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0$$

$$\therefore \ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \tag{3.5}$$

Taking the powers of e, we have

$$i(t) = I_0 e^{-Rt/L} \tag{3.6}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. The current response is shown in Fig.3.2.

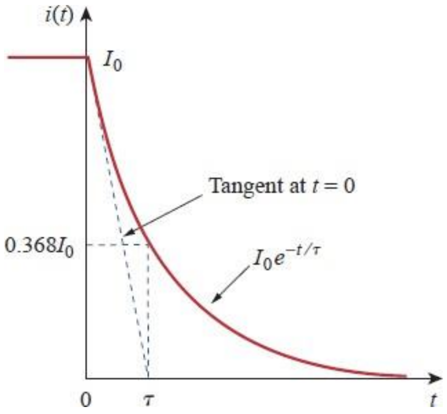


Fig.3.2. The current response of the RL circuit

It is evident from Eq. (3.6) that the time constant for the RL circuit is

$$\tau = \frac{L}{R} \text{ in (s)} \tag{3.7}$$

$$\therefore i(t) = I_0 e^{-t/\tau} \tag{3.8}$$

$$v_R(t) = iR = I_0 R e^{-t/\tau} \tag{3.9}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau} \tag{3.10}$$

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2\,R e^{-2t/\tau} dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}) \tag{3.11}$$

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$ which is the same as $w_L(0)$,the initial energy stored in the inductor as in Eq. (3.2). Again, energy initially stored in the inductor is eventually dissipated in the resistor.

With the two items, we obtain the response as the inductor current $i_L(t)=i(t) = i(0)\ e^{-t/\tau}$. Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained. **Note that in general, *R* in Eq. (3.7) is the Thevenin resistance at the terminals of the inductor.**

Example 3: Assuming that $i(0) = 10A$, calculate $i(t)$ and $i_x(t)$ in the circuit of Fig.1.

Solution:

There are two ways we can solve this problem. One way is to obtain the equivalent resistance at the inductorterminals and then use Eq. (3.8). The other way is to start from scratch by using Kirchhoff's voltage law. Whichever approach is taken, it is always better to first Obtain the inductor current.

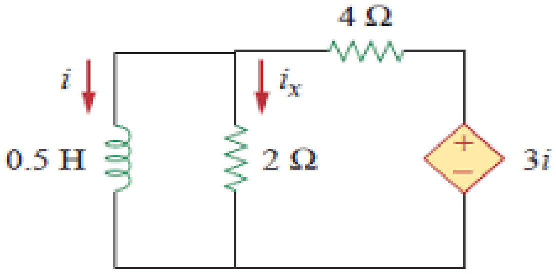


Fig.1

METHOD 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1V$ at the inductor terminals *a*-*b*, as in Fig. 2(a). (We could also insert a 1-*A* current source at the terminals.) Applying KVL to the two loops results in

$$2(i_1 - i_2) + 1 = 0 \Rightarrow i_1 - i_2 = -\frac{1}{2} \quad (1)$$

$$6i_2 - 2i_1 - 3i_1 = 0 \Rightarrow i_2 = \frac{5}{6}i_1 \quad (2)$$

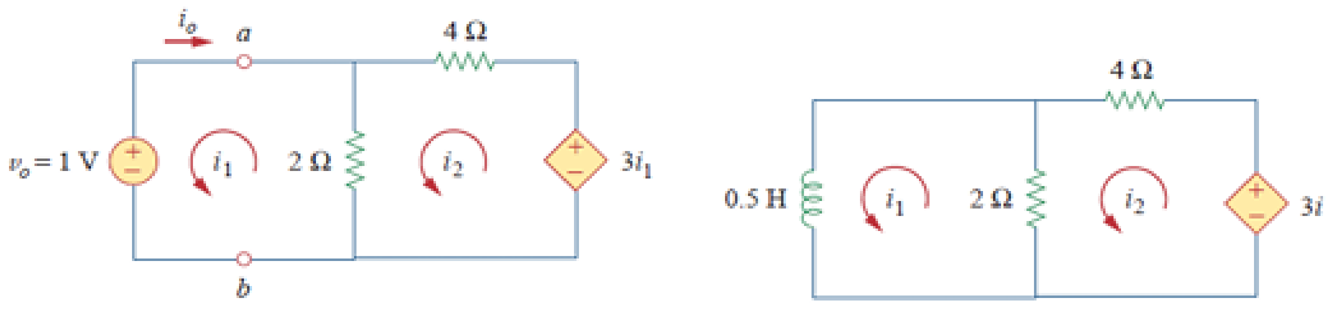
Substituting Eq. (2) into Eq. (1) gives

$$i_1 = -3A \quad i_o = -i_1 = 3A$$

$$\therefore R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}s$$

$$i(t) = i(0)e^{-\frac{t}{\tau}} = 10e^{-\left(\frac{2}{3}\right)t}A \quad t > 0$$



(a) (b)

METHOD 2 We may directly apply KVL to the circuit as in Fig. 2(b).

For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0$$

$$\therefore \frac{di_1}{dt} + 4i_1 - 4i_2 = 0 \qquad (1)$$

For loop 2,

$$6\, i_2 - 2i_1 - 3i_1 = 0 \Rightarrow i_2 = \frac{5}{6} i_1 \qquad (2)$$

Substituting Eq. (2) into Eq. (1) gives

$$\frac{di_1}{dt} + \frac{2}{3} i_1 = 0 \qquad \Rightarrow \qquad \frac{di_1}{i_1} = -\frac{2}{3} dt$$

Since $i_1 = i$, we may replace i_1 with i and integrate:

$$\ln i\Big|_{i(0)}^{i(t)} = -\frac{2}{3} t\Big|_0^t \qquad \Rightarrow \qquad \ln \frac{i(t)}{i(0)} = -\frac{2}{3} t$$

Taking the powers of e, we finally obtain

$$i(t) = i(0) e^{-\left(\frac{2}{3}\right)t} = 10 e^{-\left(\frac{2}{3}\right)t} A \qquad t > 0$$

which is the same as by **Method 1**.

$$v_L = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3} e^{-\left(\frac{2}{3}\right)t} \quad V$$

Since the inductor and the $2 - \Omega$ resistor are in parallel, $(v_L = v_R)$

$$i_x(t) = \frac{v_R}{2} = -1.6667 e^{-\left(\frac{2}{3}\right)t} A \qquad t > 0$$

H.W.3: Find i and v_x in the circuit of Fig. 1. Let $i(0) = 5$ A.

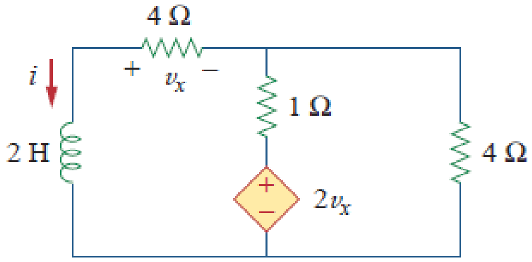


Fig. 1.

Answer: $\boldsymbol{5e^{-4t}V \quad -20e^{-4t}V}$.

Example 4: The switch in the circuit of Fig. 1. has been closed for a long time. At $t = 0$,the switch is opened. Calculate $i(t)$ for $t > 0$.

Solution:

When $t < 0$, the switch is closed, and the inductoracts as a short circuit to dc. The $16 - \Omega$ resistor is short-circuited; the resulting circuit is shown in Fig. 2(a). To get i_1 in Fig. 2(a), we combine the $4 - \Omega$ and $12 - \Omega$ resistors in parallel to get

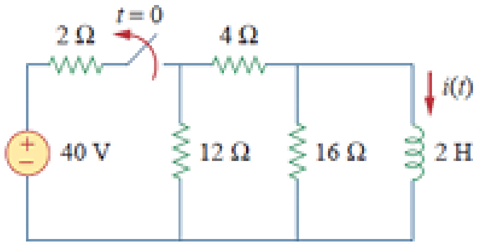


Fig. 1.

$$\frac{4 \times 12}{4+12} = 3\Omega$$

$$i_1 = \frac{40}{2+3} = 8A$$

We obtain $i(t)$ from i_1 in Fig. 2(a) using current division, by writing

$$i(t) = \frac{12}{12+4} i_1 = 6A, \quad t < 0$$

Since the current through an inductor cannot change instantaneously,

$$i(O) = i(0^-) = 6A$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free RL circuit in Fig. 2(b).

$$R_{eq} = (12+4) \| 16 = 8\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} s$$

$$\therefore i(t) = i(0)e^{-t/\tau} = 6e^{-4t} A$$

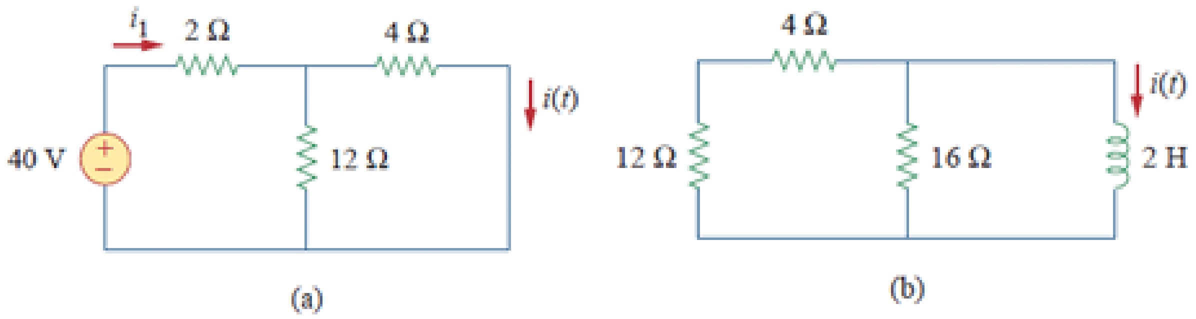


Fig. 2 Solving the circuit of Fig.1: (a) for t < 0, (b) for t > 0.

H.W.4: For the circuit in Fig.1, find $i(t)$ for $t > 0$.

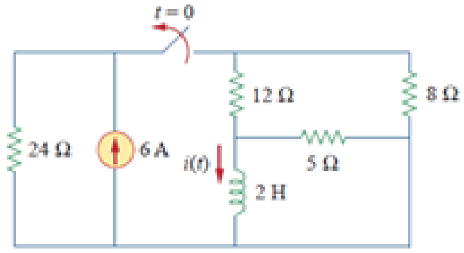


Fig.1

Answer: $2e^{-2t}$ A, t > 0

Example 5: In the circuit shown in Fig.1, find i_o, v_o, and i for all time, assuming that theswitch was open for a long time.

Solution: It is better to first find the inductor current i and then obtain other quantities from it. For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the $6-\Omega$ resistor is short-circuited, so that we have the circuit shown in Fig.2 (a). Hence,

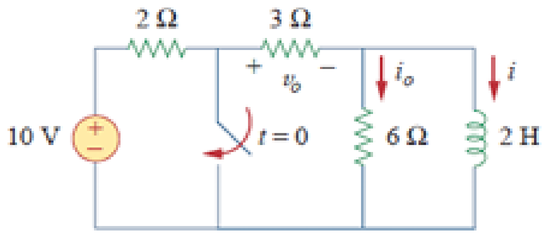


Fig.1

$$i_o = 0, \text{ and } i(t) = \frac{10}{2+3} = 2A. \quad t < 0$$

$$v_o(t) = 3i(t) = 6V. \quad t < 0$$

Thus, $i(O) = 2$.

For $t > 0$, the switch is closed, so that the voltage source is short- circuited. We now have a source-free RL circuit as shown in Fig.2 (b). At the inductor terminals,

Since the inductor is in parallel with the $6-\Omega$ and $3-\Omega$ resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}V. \quad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}A. \quad t > 0$$

Thus, for all time,

$$i_o(t) = \begin{cases} 0A & t < 0 \\ -\frac{2}{3}e^{-t}\,A & t > 0 \end{cases} \quad , \quad v_o(t) = \begin{cases} 6\,V & t < 0 \\ 4e^{-t}\,V & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2A & t < 0 \\ 2e^{-t}A & t \geq 0 \end{cases}$$

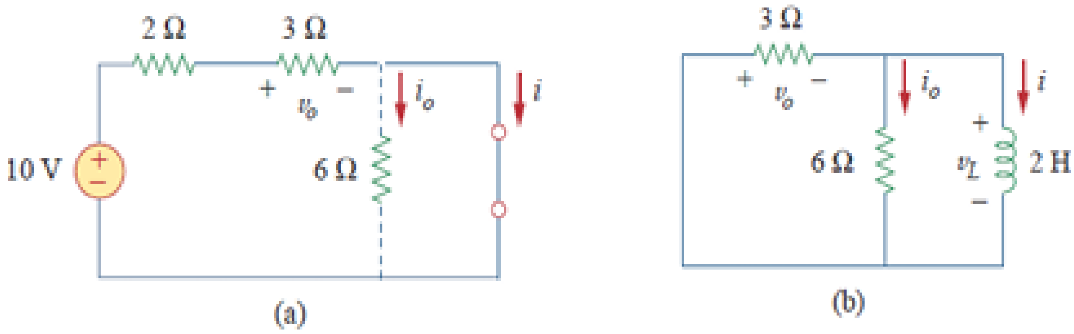


Fig.2. The circuit in Fig.1 for: (a) $t < 0$ (b) $t > 0$

We notice that the inductor current is continuous at $t = 0$, while the current through the $6 - \Omega$ resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the $3 - \Omega$ resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Fig.3. plots i and i_o.

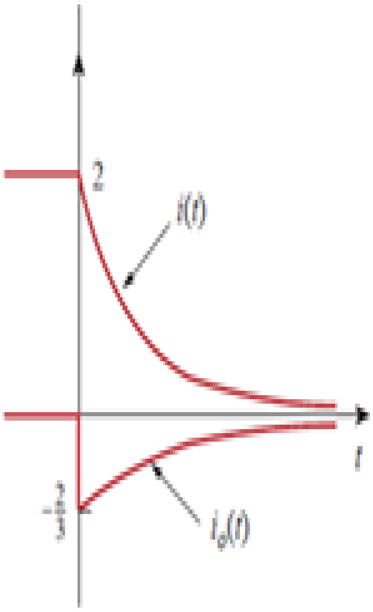


Fig.3. A plot of i and i_o

H.W.5: Determine i, i_o, and v_o for all t in the circuit shown in Fig.1. Assume that the switch was closed for a long time. It should be noted that opening a switch in series with an ideal current source creates an infinite voltage at the current source terminals. Clearly this is impossible. For the purposes of problem solving, we can place a shunt resistor in parallel with the source (which now makes it a voltage source in series with a resistor). In more practical circuits, devices that act like current sources are, for the most part, electronic circuits. These circuits will allow the source to act like an ideal current source over its operating range but voltage-limit it when the load resistor becomes too large (as inan open circuit).

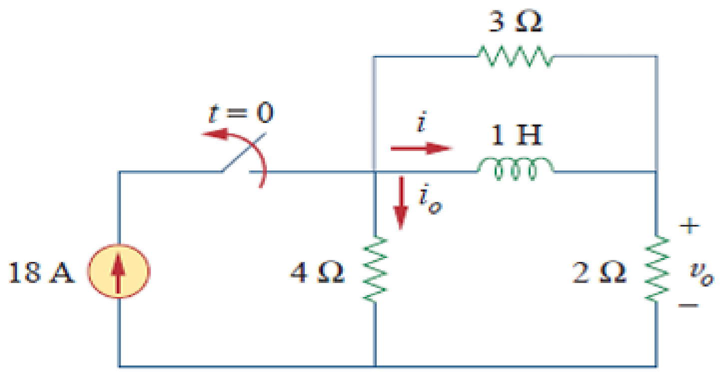


Fig.1

Answer: $i = \begin{cases} 12A & t < 0 \\ 12e^{-2t}A & t \geq 0 \end{cases}$, $i_o = \begin{cases} 6A & t < 0 \\ -4e^{-2t}A & t > 0 \end{cases}$, $v_o = \begin{cases} 24V & t < 0 \\ 8e^{-2t}V & t > 0 \end{cases}$

Thank you very much

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