Functions of Two or More variables :-

الدوال بمتغيرين أو أكثر

Before going through the functions of two or more variables lets start with the basic once,

a- Function of one variable :-

$f(x) = a$ ⟵ Function in terms of "x".

Ex) $y = 4x^2$ ⟵ Function of one variable.

y = f(x)

b- Function of two variables :-

$z = f(x, y)$ ⟵ function of two variables

Ex) $A = \frac{1}{2} bh$ ⟵ Area of the triangle is fun of two variables.

z --- dependent variable.
x, y --- independent variables.

C- Function of more than two variables :-

$w = f(x, y, z)$ ⟵ Function of three variables

w --- dep. variable.
x, y, z --- indep. variables.

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The restriction of the independent variables :-

قيود المتغيرات الغير معتمدة

Determine the domain of f :- حساب مجال/نطاق الدالة f

(1) **Ex)**

Find the domain of $f(x,y) = \ln(xy)$?

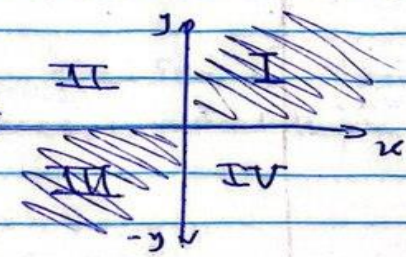
[Solution]

There are several ways to determine the fun' domain

graph — a- By graphing

as xy indep. variable under $\ln$

$xy > 0$ then all $\left\{ \begin{matrix} 0 > x > 0 \\ 0 > y > 0 \end{matrix} \right\}$



In words — b- Domain is all ordered pairs in quadrants I & III (not on axis).

Math short hand — c- Domain is all (x,y) : $\boxed{xy > 0}$

(2) **Ex)**

Find the domain of $F(x,y,z) = \dfrac{x}{\sqrt{9-x^2-y^2-z^2}}$?

[Solution]

$9 - x^2 - y^2 - z^2 > 0 \longrightarrow x^2 + y^2 + z^2 < 9$ (sphere)

a- D.: all (x,y,z) such that $x^2+y^2+z^2 < 9$.

or

In word — b- It's domain is inside a sphere or radius $r = 3$ Centered at origin $(0,0,0)$

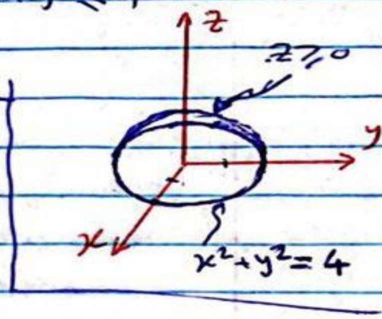
Ex) Find the domain of $F(x,y) = \sqrt{4-x^2-y^2}$ by graph?

(Solution)

$F(x,y) = z = \sqrt{4-x^2-y^2} \Longrightarrow z^2 = 4 - x^2 - y^2$

$\therefore x^2 + y^2 + z^2 = 4 \longleftarrow$ (sphere, Centered at origin with $r = 2$)

∴ As $f(x,y) = z \geq 0$, then $x^2+y^2 \leq 4$



Ex/ Find the domain by graphing of the function $f(x,y) = 1 - x - \frac{1}{2}y$

Solution

مستوي

As the Fun is of one order, the Fun is <u>plane</u>!

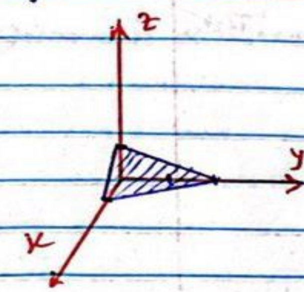
تقاطعات

So, we need to find the intercepts which is the easiest way to go with.

1- put $y = z = 0 \Longrightarrow x = 1 \longrightarrow (1,0,0)$

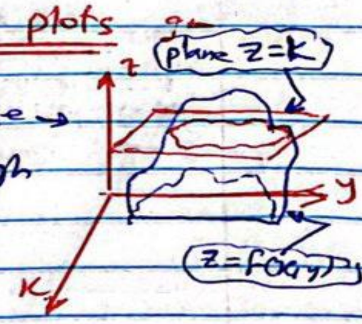
2- put $x = z = 0 \Longrightarrow y = 2 \longrightarrow (0,2,0)$

3- put $x = y = 0 \Longrightarrow z = 1 \longrightarrow (0,0,1)$



<u>Level Curves & Level Contour plots</u>

- Lets say we have some figure like →
If we take a plane & slice it through horizontally at $z = K$,
So, the intersection of a plane with a surface is called a <u>level curve</u>



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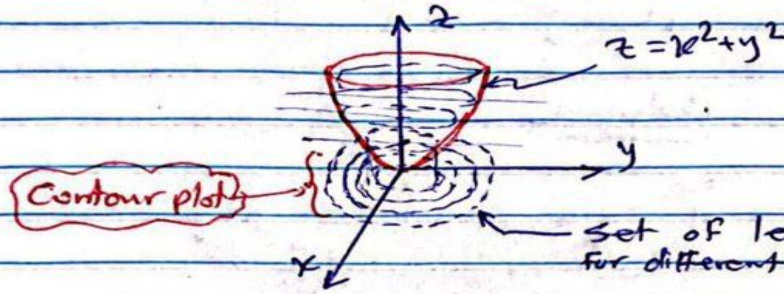
Ex) plot the Following Function & Find a level curve $z = x^2 + y^2$

Solution

1st to sketch the Fun $z = x^2 + y^2$, we need to change z & plot the Fun, from this we can Find it parabola in posative z direction.



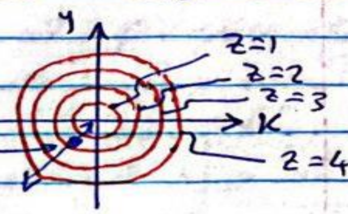
To Find a level Curve, we would give z different values, like

Put $z = 1 \Longrightarrow 1 = x^2 + y^2$

Put $z = 2 \Longrightarrow 2 = x^2 + y^2$

$\vdots \qquad \vdots \qquad \vdots$

when we sketch a set of "level Curves" we get a "Contour plot"



ح السؤال ريكون عن شكل الكنتور فيما لو اخترنا نقطة معينة وطلبنا الشكل عندما تتحرك متقربين أو مبتعدين عن نقطة الاصل .

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Ex/ Sketch the contour plot of $f(x,y)=x^2-4y^2$

Solution

لرسم الكنتور نختار عدة قيم لـ $f(x,y)$ والتي تمثل الارتفاع z، وكما في ادناه ;

* Put $z=0 \Rightarrow x^2=4y^2$

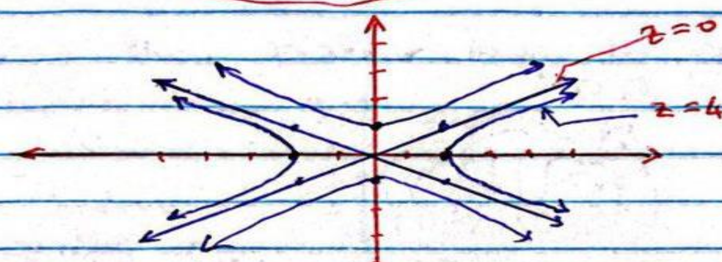
$$\therefore \boxed{x=\mp 2y} \text{ or } \boxed{y=\mp \tfrac{1}{2}x} \longrightarrow \underline{\text{line}}$$

* Put $z=4 \Rightarrow (4=x^2-4y^2)\div 4$

$$\boxed{1=\frac{x^2}{4}-y^2} \longrightarrow \text{Hyperbola}$$

* Put $z=-4 \longrightarrow (-4=x^2-4y^2)\div -4$

$$\boxed{1=y^2-\frac{x^2}{4}} \longrightarrow \text{Hyperbola}$$



هنا عندما نختار نقطة معينة على الشكل ونغير موقعها فإننا نحصل على شكل دالة سرجية (Saddle).

Level Surface :

* If we look at the graph of function of two variables, we can project a "level curve"
$F(x,y) \longrightarrow$ Two variables fun (3 Dimensions)
* But if we look at the graph of fun of three variables, we can project a "level surface"
In other way;
1. 2-variables Fun $\longrightarrow F(x,y) \longrightarrow 3D \longrightarrow$ "level curve"
2. 3-variables Fun $\longrightarrow F(x,y,z) \longrightarrow 4D \longrightarrow$ "level surface"

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Ex) consider $f(x,y,z) = x^2 + y^2 + z^2$

Here, we put $f(x,y,z) = 1 \longrightarrow 1 = x^2 + y^2 + z^2$

put $f(x,y,z) = 4 \longrightarrow 4 = x^2 + y^2 + z^2$

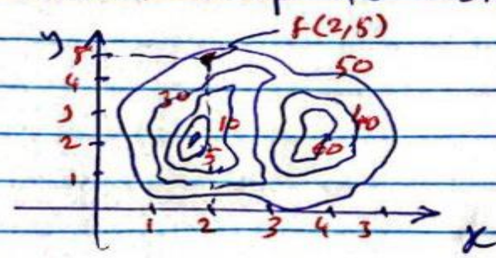
So generally;

$$K = x^2 + y^2 + z^2$$

If we plot these fun s we can get a sphere centered at the origin

So, the level surfaces are sphere centered at the origin.

Ex) Use the contour map to estimate value of $f(2,5)$



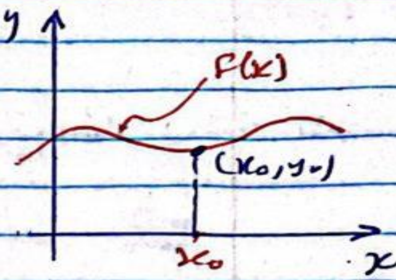
Sol.)

From the contour

$f(2,5) \approx \underline{48}$ Ans

Limits in 2D :-

Here to calculate the limit we need to calculate the limit from right side & from the left side. If they are equals, then the limit is exist.

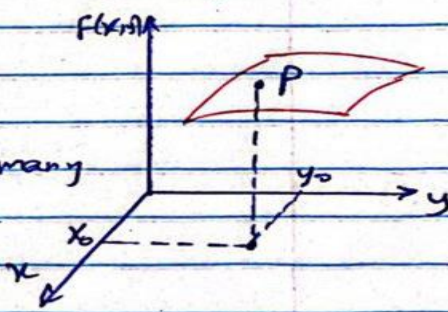


$$\lim_{x \to x_0^+} F(x) = \lim_{x \to x_0^-} F(x) \longrightarrow \text{limit exists}$$

Limits in 3D :-

If we have a plane $f(x,y)$, and we have a point $P \to (x_0, y_0)$, then when we look at the limit then we need to look at limit approaches (x_0, y_0)

$$\lim_{(x,y) \to (x_0, y_0)} f(x,y)$$



In 3-D, there are infinitely many curves approach (x_0, y_0, z_0) to check not like the 2-D which has just two (left & right) !!

ملاحظة/ كل امثلة الغايات لحالات الأكثر من بعد، هنا نحتاج ان نجرب الغاية بأكثر من طريق كأن يكون على x=0 أو y=0 أو x=y ، فاذا كانت النتيجة غير متطابقة هنا الغاية غير موجودة والعكس صحيح. ولكن بداية نحتاج ان نعوض عن قيمتي x و y بشكل مباشر ونرى إن كانت تعطي قيمة هنا الغاية موجودة أو لا فنعمل بالطريق الاول.

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Ex) Find $\lim\limits_{(x,y)\to(2,1)} f(x,y)$ if $f(x,y) = \dfrac{-xy}{x^2+y^2}$?

Sol.)

هنا نختار التعويض المباشر ونرى فيما لو يوجد حل أم لا!

$$\lim_{(x,y)\to(2,1)} \frac{-xy}{x^2+y^2} = \frac{-2*1}{(2)^2+(1)^2} = \boxed{\frac{-2}{5}} \quad \text{Ans}$$

Ex) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{-xy}{x^2+y^2}$?

Sol.)

هنا عند التعويض المباشر نلاحظ عدم وجود حل، فنختار الطريق الثاني للحل وهو باختيار أكثر من طريق ونرى النتيجة فإذا كانت متشابهة فيكون يوجد غاية أما إذا لا فلا توجد غاية.

ⓐ lim along x-axis → $\boxed{y=0}$ so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{0}{x^2} = \boxed{0}$$

ⓑ lim along y-axis → $\boxed{x=0}$

$$\lim_{(x,y)\to 0} f(x,y) = \lim_{y\to 0} \frac{0}{y^2} = \boxed{0}$$

ⓒ lim along $\boxed{y=x}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{-x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{-x^2}{2x^2} = \boxed{\frac{-1}{2}}$$

هنا بما أن الغاية على x-axis و y-axis و x=y غير متساوية بالكامل هنا ربما الغاية تكون غير موجودة.

- So, here the limit may be does not exist.

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Important Theorem :-

(a) If $f(x,y) \to L$ as $f(x,y) \to (x_0,y_0)$, then $f(x,y) \to L$ as $(x,y) \to (x_0,y_0)$ along any smooth curve.

(b) If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist (DNE) along some smooth curve or if $f(x,y)$ has different values along different curves, then the limit DNE.

Ex) Find $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$?

Sol)

By direct plug in on x & y by $(1,2)$

$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2} = \frac{10}{5} = \boxed{2} \longrightarrow \text{limit does exist}$$

Ex) Find $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$?

Sol.)

The direct substitution / plug in of x,y does not work, so lets check the limits on some paths.

(a) along $x=0$:-

$$\lim_{y\to 0} \left(\frac{-y^2}{y^2}\right)^2 = \boxed{1}$$

(different values) $\Longrightarrow$ limit DNE

(b) along $y=x$:-

$$\lim_{y\to 0} \left(\frac{y^2-y^2}{y^2+y^2}\right)^2 = \lim_{y\to 0} \left(\frac{0}{2y^2}\right)^2 = \boxed{0}$$

Continuity :- الاستمرارية

Def. : تعريف

A Function $f(x,y)$ is said to be continuous at (x_0,y_0), if $f(x_0,y_0)$ is defined &

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Note /

the function is continuous on disk D, If it is continuous at every point in an open set.

Theorem : نظرية

ⓐ $f(x,y) = g(x)\cdot h(y)$ is continuous at (x_0,y_0) if $g(x)$ is continuous at x_0 & $h(y)$ is continuous at y_0.

ⓑ Compositions are continuous $(f(x,y) = g(h(x,y))$ if $h(x,y)$ is continuous at (x_0,y_0) & $g(u)$ is continuous at $u = h(x_0,y_0)$

Ex/ Check whether the Fun $f(x,y) = 3x^2y^5$ is continuous or not ?

Sol./

From theorem ⓐ : $f(x,y) = g(x)\,h(y)$

Consider ; $\left.\begin{array}{l} g(x) = 3x^2 \\ h(y) = y^5 \end{array}\right\} \longrightarrow$ Both are continuous everywhere,

Then $f(x,y)$ is continuous everywhere

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Ex/ Is the f.n $f(x,y) = \sin(3x^2y^5)$ cont. or not?

Sol./

- $h(x,y) = 3x^2y^5$ is continuous everywhere
- $g(u) = \sin(u)$ is " "

$\therefore f(x,y)$ is cont. everywhere.

Ex/ check against continuity the Fn

$$\lim_{(x,y)\to(0,0)} y \ln(x^2+y^2)$$

Sol./

Here we can't do direct substitute on x & y. So we need to convert it to polar form as;

$r^2 = x^2+y^2$, $y = r\sin\theta$

$$\therefore \lim_{r\to 0} (r\sin\theta) \ln r^2 = \lim_{r\to 0} 2r\sin\theta \ln r$$

$$= \lim_{r\to 0} \frac{\sin\theta \ln r}{\frac{1}{2r}} = \frac{\infty}{\infty} \quad \text{(L'Hopital Rule)}$$

$$= \lim_{r\to 0} \frac{\sin\theta \cdot \frac{1}{r}}{\frac{-1}{2r^{2}}} = \lim_{r\to 0} (\sin\theta * (-2r))$$

$= \boxed{0}$ Here the limit does exist even though we can't plug in (0,0) !!

Partial Derivatives : المشتقات الجزئية

Definition : تعريف

If $z = f(x,y)$, then the 1st derivative with respect to x & y are partial derivative of x (f_x) & partial derivative of y (f_y) respectively, defined by;

$$f_x = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Ex1) Find f_x & f_y of $f(x,y) = 3x - x^2y^2 + 2x^3y$

Sol.)

$$f_x = \frac{\partial f}{\partial x} = 3 - 2xy^2 + 6x^2y \quad \text{(treat } y \text{ like a Const.)}$$

$$f_y = \frac{\partial f}{\partial y} = -2x^2y + 2x^3 \quad \text{(treat } x \text{ like a constant)}$$

Ex2) If $f(x,y) = x\,e^{x^2y}$, Find f_x, f_y and evaluate f_x & f_y at $(1, \ln 2)$?

Solution)

- $f_x = x \cdot 2xy\,e^{x^2y} + e^{x^2y} \longrightarrow f_x\big|_{(1,\ln 2)} = 2\ln 2\, e^{\ln 2} + e^{\ln 2} = \boxed{4\ln 2 + 2}$

- $f_y = x \cdot x^2\,e^{x^2y} = x^3 e^{x^2y} \longrightarrow f_y\big|_{(1,\ln 2)} = 1^3 \cdot e^{1^2 \cdot \ln 2} = \boxed{2}$

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Higher Order Partial Derivative :-

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

  x is 1st, then y.

Ex) Find f_{xx}, f_{yy}, f_{xy} & f_{yx} for

$f(x,y) = 3xy^2 - 2y + 5x^2y^2$

Sol.)

$f_x = 3y^2 + 10xy^2$; $f_y = 6xy - 2 + 10x^2y$

$f_{xx} = 10y^2$; $f_{yy} = 6x + 10x^2$

$f_{xy} = 6y + 20xy$; $f_{yx} = 6y + 20xy$

Ex) If $z = x^3 + y^4 + x\sin y + y\cos x$ then find z_{xy}, z_{yx}

Sol.)

- $z_x = 3x^2 + \sin y - y\sin x$
- $z_{yx} = \frac{\partial}{\partial y}(z_x) = \cos y - \sin x$
- $z_y = 4y^3 + x\cos y + \cos x$
- $z_{xy} = \frac{\partial}{\partial x}(z_y) = \cos y - \sin x$

Note) From these two examples, we can see that $f_{xy} = f_{yx}$ & $z_{xy} = z_{yx}$!!

Theorem : نظرية

If f is a fun of x & y such that f_{xy} & f_{yx} are continuous on an open disk R, then for every (x,y) in R, $f_{xy}(x,y) = f_{yx}(x,y)$.

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Ex/ If $f(x,y) = y \cdot \sin(xy)$, then Find f_x & f_y ?

Sol/

$f_x = \frac{\partial f}{\partial x} = y * \cos(xy) * y + \sin(xy) * 0 = \boxed{y^2 \cos(xy)}$

$f_y = \frac{\partial f}{\partial y} = y \cdot \cos(xy) \cdot x + \sin(xy) * 1$
$= \boxed{yx \cos(xy) + \sin(xy)}$

Ex/ If $z = \tan^{-1}\left(\frac{y}{x}\right)$ then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Sol/

$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{x*0 - y*1}{x^2} = \boxed{\frac{-y}{x^2+y^2}}$

$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{x*1 - y*0}{x^2} = \boxed{\frac{x}{x^2+y^2}}$

$\therefore x\left(\frac{-y}{x^2+y^2}\right) + y\left(\frac{x}{x^2+y^2}\right) = \boxed{0}$ ✓

Question For Discussion :-

If $f(x,y,z) = y e^{x} + x \ln z$ show that:
$f_{xzz} = f_{zxz} = f_{zzx}$

Functions of 2 and More Variables, Dept and Indept Variables, Limits, Continuity, Partial Derivatives -- نهاية محاضرة "دوال بمتغيرين وأكثر، متغيرات معتمدة وغير معتمدة، الغايات، الأستمرارية، المشتقات الجزئية"--

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