Discrete Mathematics

Lecture 3

Predicates and Quantifiers

By

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Introduction

- Propositional logic, studied in previous lectures cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

"Every computer connected to the university network is functioning properly"

- No rules of propositional logic allow us to conclude the truth of the statement.
- In this lecture we will introduce a more powerful type of logic called ***predicate logic.***

Predicate:

x is greater than 3

Predicate:

$$\underbrace{\boxed{x}}_{\text{The subject}}\ \underbrace{\boxed{\text{is greater than 3}}}_{\text{The predicate}}$$

We can denote the statement "x *is greater than* 3" by $\boldsymbol{P(x)}$

where $\boldsymbol{P}$ denotes the predicate "*is greater than* 3" and $\boldsymbol{x}$ is the variable.

The statement $\boldsymbol{P(x)}$ is also said to be the value of the **propositional function** $\boldsymbol{P}$ at $\boldsymbol{x}$. Once a value has been assigned to the variable $\boldsymbol{x}$, the statement $\boldsymbol{P(x)}$ becomes a proposition and has a truth value.

Example1:

Let $P(x)$ denote the statement "$x > 3$."

What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement *P(4)* by setting $x = 4$ in the statement "$x > 3$." Hence, *P(4),* which is the statement "$4 > 3$," is true. However, *P(2),* which is the statement "$2 > 3$," is false.

Example1:

Let $P(x)$ denote the statement "$x > 3$."

What are the truth values of $P(4)$ and $P(2)$?

T F

Example2:

Let $Q(x, y)$ denote the statement "$x = y + 3$."

What are the truth values of the propositions

$Q(1, 2)$ and $Q(3, 0)$?

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Example3:

1. **1.** Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?

   a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$

2. **2.** Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?

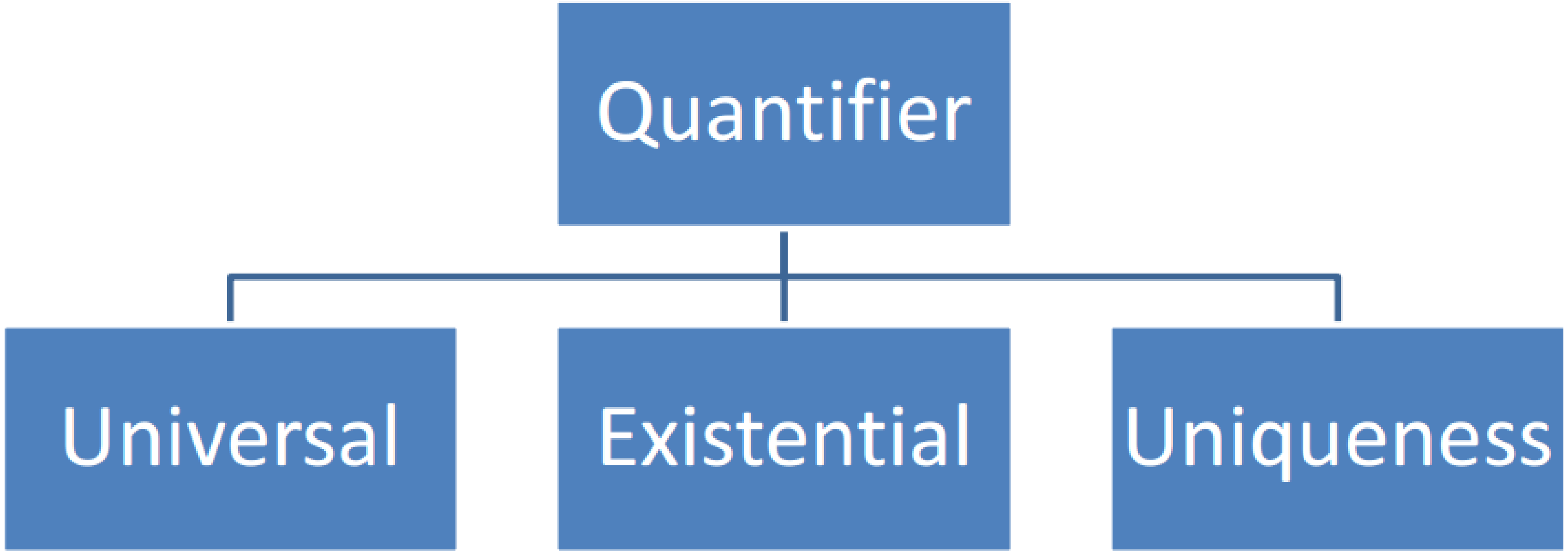
   a) $P(\text{orange})$ **b)** $P(\text{lemon})$
   c) $P(\text{true})$ **d)** $P(\text{false})$

Example3:

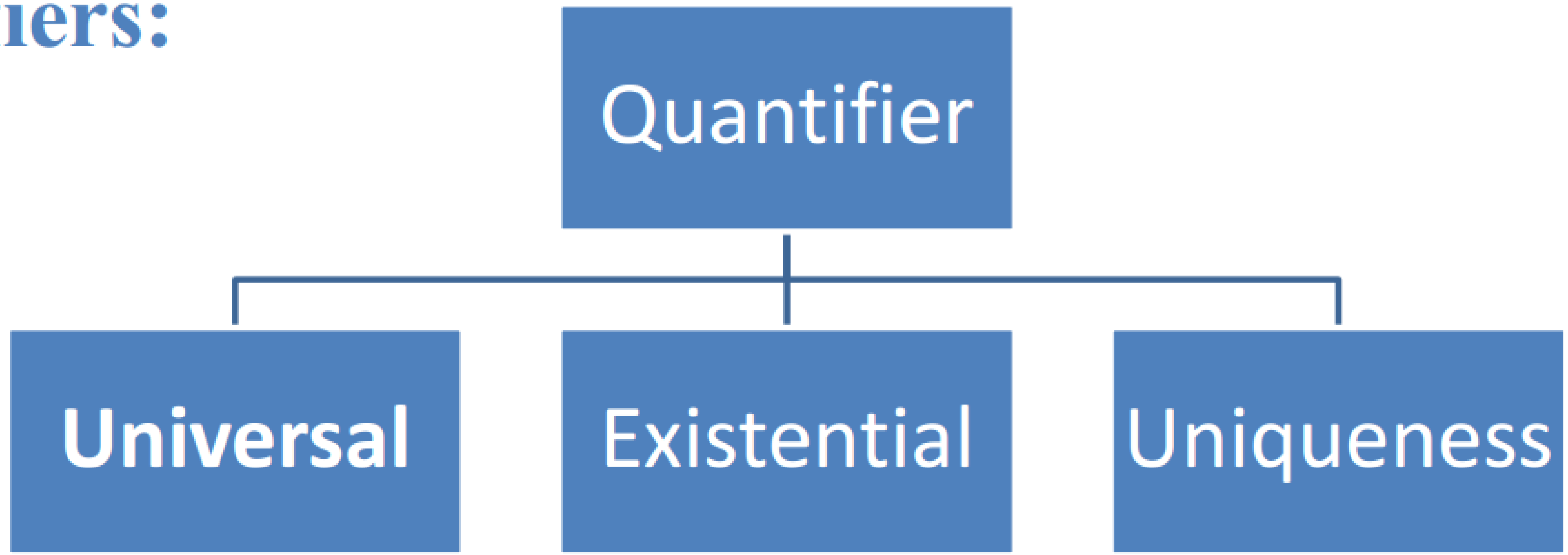
1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?
   a) $P(0)$ T b) $P(4)$ T c) $P(6)$ F
2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?
   a) $P(\text{orange})$ T b) $P(\text{lemon})$ F
   c) $P(\text{true})$ F d) $P(\text{false})$ T

Quantifiers:

Expresses the extent to which a predicate is true over a **range** of elements.



Quantifiers:

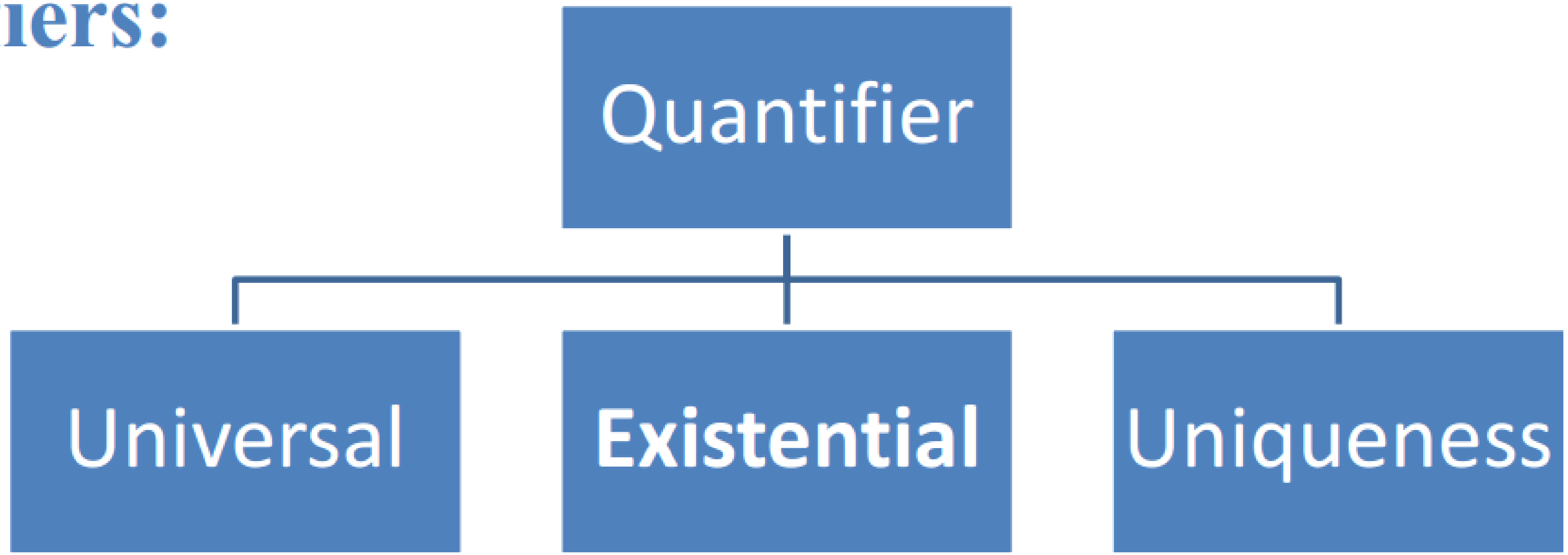


$\forall x P(x)$

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

Quantifiers:

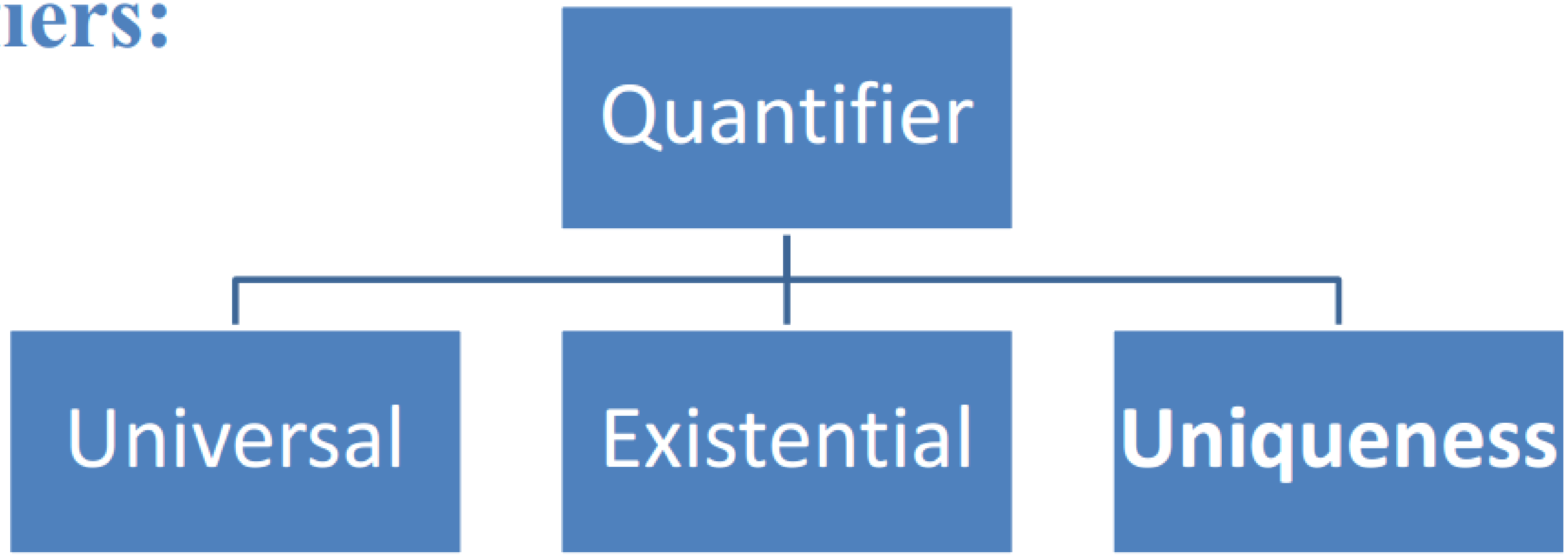


$$\exists x \, P(x)$$

The *existential quantification* of $P(x)$ is the proposition

“There exists an element x in the domain such that $P(x)$.”

Quantifiers:

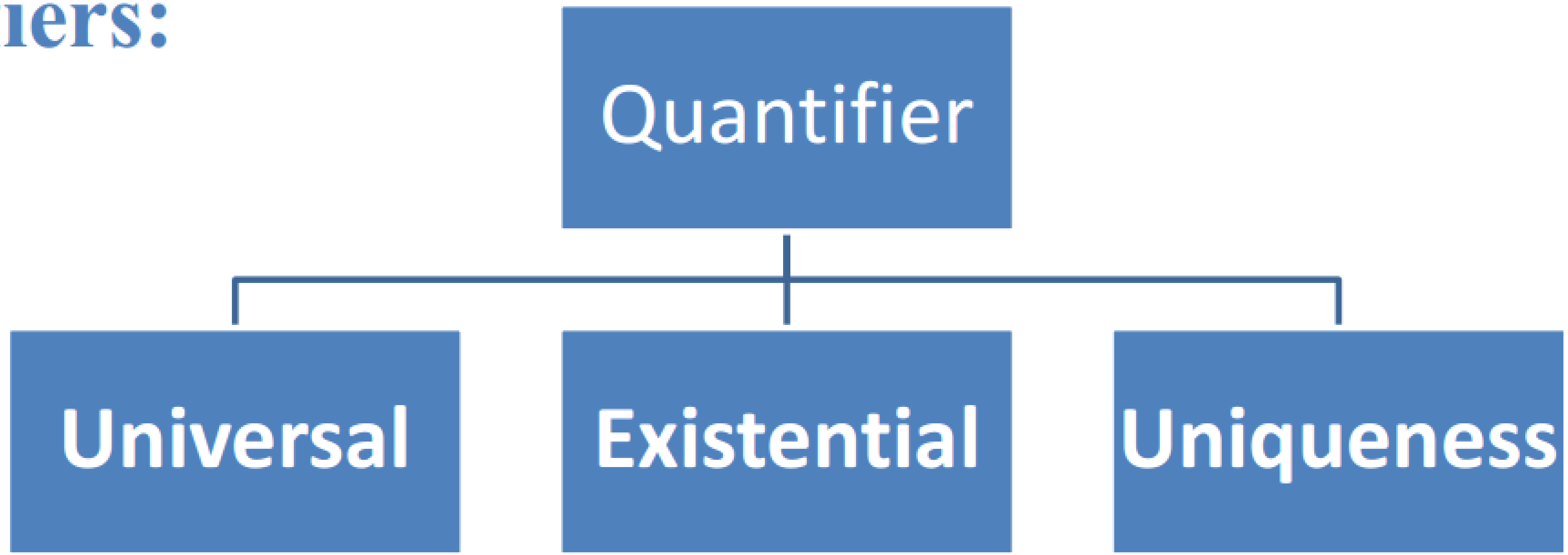


$$\exists!\, xP(x)$$

$$\exists_1 xP(x)$$

"There exists a unique x such that $P(x)$ is true."

Quantifiers:



$\forall x P(x)$ $\exists x P(x)$ $\exists ! xP(x),\ \exists_1 xP(x)$

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example1:

Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

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Let $P(x)$ be the statement "$x + 1 > x$."

What is the truth value of the quantification $\forall x P(x)$,

where the domain consists of all real numbers?

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x)$$

is true.

Example2:

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

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Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus $\forall x\, Q(x)$ is false.

Example3:

Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

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Let $P(x)$ denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$,

where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Example 4:

What is the truth value of ∀x P(x), where P(x) is the statement “$x^2 < 10$” and the domain consists of the positive integers not exceeding 4?

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What is the truth value of ∀x P(x), where P(x) is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution:

The statement ∀x P(x) is the same as the conjunction

$$P(1) \land P(2) \land P(3) \land P(4)$$

Because the domain consists of the integers 1, 2, 3, and 4. Because P(4), which is the statement "$4^2<10$," is false, it follows that ∀x P(x)is false.

Example5:

What is the truth value of $\exists x P(x)$,

where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$,

the proposition $\exists x P(x)$ is the same as the disjunction

$P(1) \vee P(2) \vee P(3) \vee P(4)$.

Because $P(4)$, which is the statement "$4^2 > 10$," is true,

it follows that $\exists x P(x)$ is true.

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ **b)** $P(1)$ **c)** $P(2)$
d) $P(-1)$ **e)** $\exists x P(x)$ **f)** $\forall x P(x)$

Example6:

Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ T **b)** $P(1)$ T **c)** $P(2)$ F

d) $P(-1)$ F **e)** $\exists x P(x)$ T **f)** $\forall x P(x)$ F

Homework 1

Let Q(x, y)denote the statement “x is the capital of y.”

What are these truth values

a) Q(Denver, Colorado)

b)Q(Detroit, Michigan)

c) Q(Massachusetts, Boston)

d)Q(New York, New York)

Homework 2

Let P(x)be the statement “x spends more than five hours every weekday in class,” where the domain for x consists of all students. Express each of these quantifications in English

a) $\exists x\, P(x)$

b) $\forall x\, P(x)$

c) $\exists x\, \neg P(x)$

d) $\forall x\, \neg P(x)$