جامـــعة المـــستـقبـل
AL MUSTAQBAL UNIVERSITY

Discrete Mathematics

Lecture 3

Propositional Equivalences

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Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term “compound proposition” to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology .

A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contra diction is called a contingency.

EXAMPLE 1

We can construct examples of tautologies and contradictions using just one propositional variable.

Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction.

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Logical Equivalences

DEFINITION 2

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark:

The symbol ≡ is not a logical connective, and $p \equiv q$ is not a compound proposition

but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol⇔is sometimes used insteadof ≡ to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truthtable.

In particular, the compound propositions p and q are equivalent if and only if the columns

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Using De Morgan's Laws

The two logical equivalences known as De Morgan's laws are particularly important. They tell When using De Morgan's laws, remember to change

the logical connective after you negate. us how to negate conjunctions and how to negate disjunctions. In particular.

the equivalence

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ tells us that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions. Similarly, the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$ tells us that the negation of a conjunction is formed by taking the disjunction of the negations of the component propositions. Example 5 illustrates the use of De Morgan's laws.

EXAMPLE 5

Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution:

Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.

Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer ."Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert."

<u>Constructing New Logical Equivalences</u>

The logical equivalences in Table 6, as well as any others that have been established (such as those shown in Tables 7 and 8), can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition. This technique is illustrated in Examples 6–8, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent (see Exercise 56).

EXAMPLE 6

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

We could use a truth table to show that these compound propositions are equivalent (similar to what we did in Example 4). Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 6 at a time, starting with $\neg(p \to q)$ and ending with $p \land \neg q$. We have the following equivalences.

$\neg(p \to q) \equiv \neg(\neg p \lor q)$ by Example 3 $\equiv \neg(\neg p) \land \neg q$ by the second De Morgan law $\equiv p \land \neg q$ by the double negation law