Functions of Two or More variables :-

الدوال بمتغيرين أو أكثر

Before going through the functions of two or more variables lets start with the basic once,

a- Function of one variable :-

$f(x) = a$ ← Function in terms of "x".

Ex) $y = 4x^2$ ← Function of one variable.

$f(x)$

b- Function of two variables :-

$Z = f(x, y)$ ← function of two variables

Ex) $A = \frac{1}{2} bh$ ← Area of the triangle is fun of two variables.

z --- dependent variable.
x, y --- independent variables.

c- Function of more than two variables :-

$W = f(x, y, z)$ ← Function of three variables

w --- dep. variable.
x, y, z --- indep. variables.

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The restriction of the independent variables :-

قيود المتغيرات الغير معتمدة

Determine the domain of f :- f حساب مجال/نطاق الدالة

(A) **Ex1**

Find the domain of $f(x,y) = \ln(xy)$?

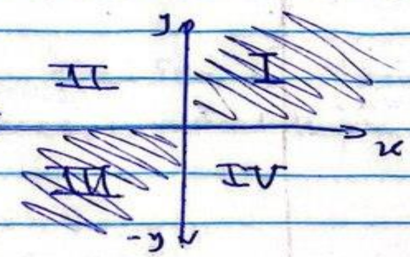
[Solution]

There are several ways to determine the fun' domain

graph — a- By graphing

as xy indep. variable under $\ln$

$xy > 0$ then all $\left\{ \begin{array}{l} 0 > x > 0 \\ 0 > y > 0 \end{array} \right.$?



In words — b- Domain is all ordered pairs in quadrants I & III (not on axis).

Math short hand — c- Domain is all (x,y) : $\boxed{xy > 0}$

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(B) **Ex)**

Find the domain of $F(x,y,z) = \dfrac{x}{\sqrt{9-x^2-y^2-z^2}}$?

[Solution]

$9 - x^2 - y^2 - z^2 > 0 \longrightarrow x^2 + y^2 + z^2 < 9$ (sphere)

a- D. : all (x,y,z) such that $x^2+y^2+z^2 < 9$.

or

In word — b- It's domain is inside a sphere of radius $r = 3$ Centered at origin $(0,0,0)$

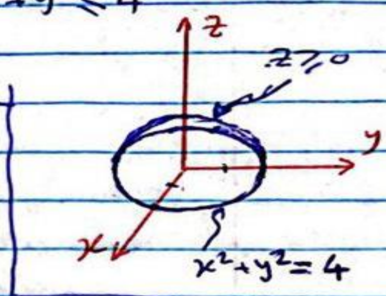
Ex) Find the domain of $f(x,y) = \sqrt{4-x^2-y^2}$ by graph?

(Solution)

$F(x,y) = z = \sqrt{4-x^2-y^2} \longrightarrow z^2 = 4 - x^2 - y^2$

$\therefore x^2 + y^2 + z^2 = 4 \longleftarrow$ (sphere, centered at origin with $r = 2$)

As $f(x,y) = z \geq 0$, then $x^2+y^2 \leq 4$

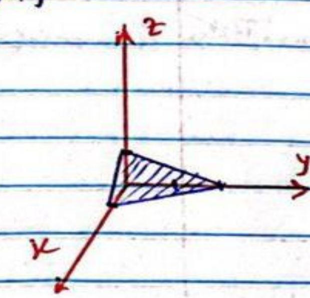


Ex/ Find the domain by graphing of the function $f(x,y) = 1 - x - \frac{1}{2}y$

Solution

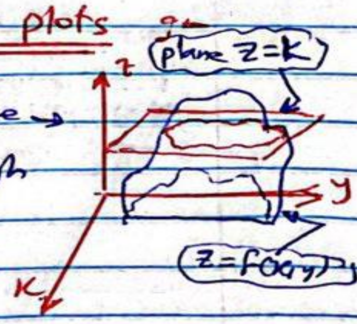
As the Fun is of one order, the Fun is plane!
So, we need to find the intercepts which is the easiest way to go with.

1- put $y=z=0 \Longrightarrow x = 1 \longrightarrow (1,0,0)$
2- put $x=z=0 \Longrightarrow y = 2 \longrightarrow (0,2,0)$
3- put $x=y=0 \Longrightarrow z = 1 \longrightarrow (0,0,1)$



Level Curves or Level Contour plots

Lets say we have some figure like →
If we take a plane & slice it through horizontally at $z=K$,
So, the intersection of a plane with a surface is called a level curve



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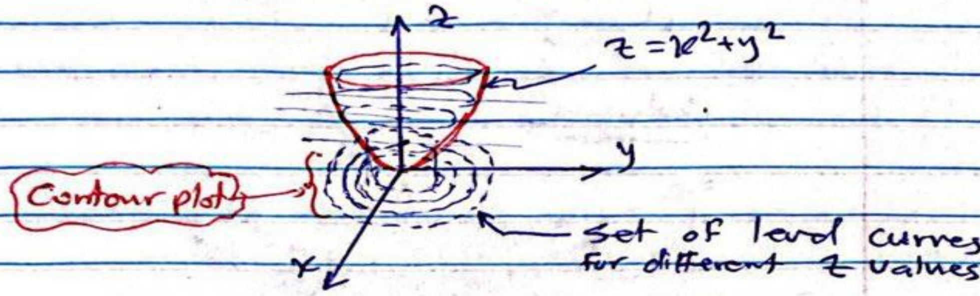
Ex/ plot the Following Function & Find a level curve $z = x^2 + y^2$

Solution/

1st to sketch the Fun $z = x^2 + y^2$, we need to change z & plot the Fun, from this we can Find it parabola in posative z direction.



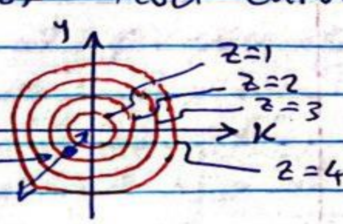
To Find a level Curve, we would give z different values, like

put $z = 1 \Longrightarrow 1 = x^2 + y^2$

put $z = 2 \Longrightarrow 2 = x^2 + y^2$

$\vdots \quad \vdots \quad \vdots$

when we sketch a set of "level Curves" we get a "Contour plot"



السؤال يكون عن شكل الكنتور فيما لو اخترنا نقطة معينة وطبعا الشكل عندما تتحرك تقترب أو مبتعدين عن نقطة الاصل .

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Ex/ sketch the contour plot of $f(x,y) = x^2 - 4y^2$

Solution

لرسم الكنتور نختار عدة قيم لـ $f(x,y)$ والتي تمثل الارتفاع z، وكما في ادناه ؛

* put $z = 0 \Rightarrow x^2 = 4y^2$

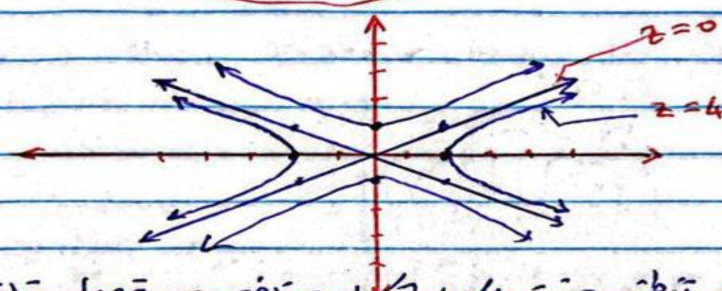
$\therefore$ or $\boxed{x = \mp 2y}$ / $\boxed{y = \mp \frac{1}{2} x}$ $\longrightarrow$ line

* put $z = 4 \Rightarrow (4 = x^2 - 4y^2) \div 4$

$\boxed{1 = \frac{x^2}{4} - y^2} \longrightarrow$ Hyperbola

* put $z = -4 \longrightarrow (-4 = x^2 - 4y^2) \div -4$

$\boxed{1 = y^2 - \frac{x^2}{4}} \longrightarrow$ Hyperbola



هنا عندما نختار نقطة معينة على الشكل ونغير موقعها تأخذ شكل على شكل دالة سرجية (Saddle).

Level Surface :

* If we look at the graph of function of two variables, we can project a "level curve"

$F(x,y) \longrightarrow$ Two variables fun (3 Dimensions)

* But if we look at the graph of fun of three variables, we can project a "level surface"

In other way;

① 2-variables Fun $\longrightarrow F(x,y) \longrightarrow 3D \longrightarrow$ "level curve"

② 3-variables Fun $\longrightarrow F(x,y,z) \longrightarrow 4D \longrightarrow$ "level surface"

Ex) consider $f(x,y,z) = x^2 + y^2 + z^2$

Here, we put $f(x,y,z) = 1 \longrightarrow 1 = x^2 + y^2 + z^2$

put $f(x,y,z) = 4 \longrightarrow 4 = x^2 + y^2 + z^2$

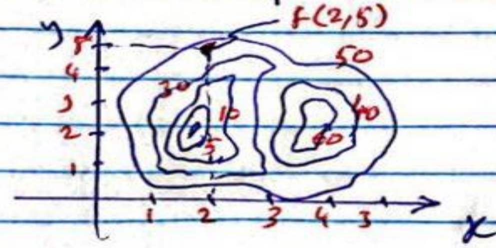
So generally;

$$K = x^2 + y^2 + z^2$$

If we plot these forms we can get a sphere centered at the origin

So, the level surfaces are sphere centered at the origin.

Ex) Use the contour map to estimate value of $f(2,5)$



Sol.)

From the contour

$f(2,5) \approx \underline{48}$ Ans

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Partial Derivatives : المشتقات الجزئية

Definition : تعريف

If $z = f(x,y)$, then the 1st derivative with respect to x & y are partial derivative of f of x (f_x) & partial derivative of y (f_y) respectively. defined by;

$$f_x = \frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y = \frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Ex) Find f_x & f_y of $f(x,y) = 3x - x^2y^2 + 2x^3y$

Sol.)

$f_x = \frac{\partial f}{\partial x} = 3 - 2xy^2 + 6x^2y$ (treat y like a Const.)

$f_y = \frac{\partial f}{\partial y} = -2x^2y + 2x^3$ (treat x like a constant)

Ex) If $f(x,y) = x e^{x^2y}$, Find f_x, f_y and evaluate f_x & f_y at $(1, \ln 2)$?

Solution)

$f_x = x \cdot 2xy\, e^{x^2y} + e^{x^2y} \longrightarrow f_x\big|_{(1,\ln 2)} = 2\ln 2\, e^{\ln 2} + e^{\ln 2}$

$= \boxed{4\ln 2 + 2}$

$f_y = x \cdot x^2 e^{x^2y} = x^3 e^{x^2y} \longrightarrow f_y\big|_{(1,\ln 2)} = 1^3 \cdot e^{1^2 \cdot \ln 2} = \boxed{2}$

Higher Order Partial Derivative :-

- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

  x is 1st, then y.

Ex) Find f_{xx}, f_{yy}, f_{xy} & f_{yx} for
$f(x,y) = 3xy^2 - 2y + 5x^2y^2$

Sol.)

$f_x = 3y^2 + 10xy^2$; $f_y = 6xy - 2 + 10x^2y$
$f_{xx} = 10y^2$; $f_{yy} = 6x + 10x^2$
$f_{xy} = 6y + 20xy$; $f_{yx} = 6y + 20xy$

Ex) If $z = x^3 + y^4 + x\sin y + y\cos x$ then find z_{xy}, z_{yx}

Sol.)

- $z_x = 3x^2 + \sin y - y\sin x$
- $z_{yx} = \frac{\partial}{\partial y}(z_x) = \cos y - \sin x$
- $z_y = 4y^3 + x\cos y + \cos x$
- $z_{xy} = \frac{\partial}{\partial x}(z_y) = \cos y - \sin x$

Note) From these two examples, we can see that $f_{xy} = f_{yx}$ & $z_{xy} = z_{yx}$!!

Theorem : نظرية

If f is a fun of x & y such that f_{xy} & f_{yx} are continuous on an open disk R, then for every (x,y) in R, $f_{xy}(x,y) = f_{yx}(x,y)$.

Ex/ If $f(x,y) = y \cdot \sin(xy)$, then Find f_x & f_y?

Sol/

$f_x = \frac{\partial f}{\partial x} = y * \cos(xy) * y + \sin(xy) * 0 = \boxed{y^2 \cos(xy)}$

$f_y = \frac{\partial f}{\partial y} = y \cdot \cos(xy) * x + \sin(xy) * 1$

$= \boxed{yx \cos(xy) + \sin(xy)}$

Ex/ If $z = \tan^{-1}(\frac{y}{x})$ then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Sol/

$\frac{\partial z}{\partial x} = \frac{1}{1+(\frac{y}{x})^2} * \frac{x*0 - y*1}{x^2} = \boxed{\frac{-y}{x^2+y^2}}$

$\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{y}{x})^2} * \frac{x*1 - y*0}{x^2} = \boxed{\frac{x}{x^2+y^2}}$

$\therefore x\left(\frac{-y}{x^2+y^2}\right) + y\left(\frac{x}{x^2+y^2}\right) = \boxed{0}$ ✓

Question For Discussion :-

If $f(x,y,z) = y e^{x} + x \ln z$ show that:

$f_{xzz} = f_{zxz} = f_{zzx}$

Chain Rule for Partial Derivatives:-

① If $w = f(x,y)$, $x = g(u)$ & $y = h(u)$, then:

$$\frac{dw}{du} = \frac{\partial w}{\partial x} \cdot \frac{dx}{du} + \frac{\partial w}{\partial y} \cdot \frac{dy}{du}$$

(2) If $w = f(x,y)$, $x = g(u,v)$ & $y = h(u,v)$, Then

$$\frac{dw}{du} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{dw}{dv} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Ex) Use the chain Rule to find the derivative of $f(x,y) = xy$ with respect to t along the path $x = \cos t$ & $y = \sin t$?

Solution:

$\frac{\partial f}{\partial x} = y = \sin t$; $\frac{dx}{dt} = -\sin t$

$\frac{\partial f}{\partial y} = x = \cos t$; $\frac{dy}{dt} = \cos t$

$$\therefore \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \sin t\cdot(-\sin t) + \cos t\cdot\cos t$$

$$= \boxed{\cos^2 t - \sin^2 t} = \boxed{\cos 2t}$$

Ex) Find the value of $\frac{df}{dt}$ at $t=0$ if $f(x,y,z) = xy + z$ & $x = \cos t$, $y = \sin t$, $z = t$

Solution:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \overset{\sin t}{\cancel{y}}\cdot(-\sin t) + \overset{\cos t}{\cancel{x}}\cdot\cos t + 1 * 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\left.\frac{df}{dt}\right|_{t=0} = \cos(0) + 1 = 1 + 1 = \boxed{2}$$ Ans

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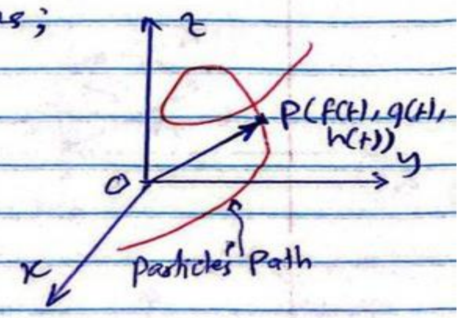
Vector Valued Functions:-

From lecture #2, we introduced a Parametric line or curve, which represent a partical moves through space during a time interval or parameter "t", these Fnns are defined as;

$x = f(t)$
$y = g(t)$
$z = h(t)$



when time t increases the point $p(x,y,z) = (f(t), g(t), h(t))$ make up a curve in space called the "particle's path"

A curve in space can also be represented in vector form, this form called a vector function or a vector valued function

$$\vec{r}(t) = \vec{OP} = f(t)\,i + g(t)\,j + h(t)\,k$$

$f(t), g(t), h(t)$ } ⟵ "Component function of $\vec{r}(t)$"

So, at any given time t value, $\vec{r}(t)$ represents a vector whose initial point is at the origin & terminal point is $(f(t), g(t), h(t))$

Domain = All real numbers $\mathbb{R}$ -
Range : Set of vectors .

Graph of Vector Value Function:-
It is the curve that traced by connecting tips of "radius" vectors $\vec{r}(t)$.

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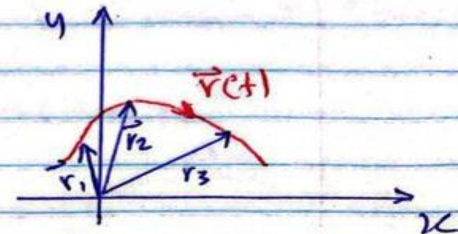
Ex) lets say we have $\vec{r}_1$ at $t=0$, & $\vec{r}_2$ at $t=1$ & $\vec{r}_3$ at $t=2$, all what we need is to graph these three vectors & connect the arrows tips by a curve, this curve represents a graph of vector function

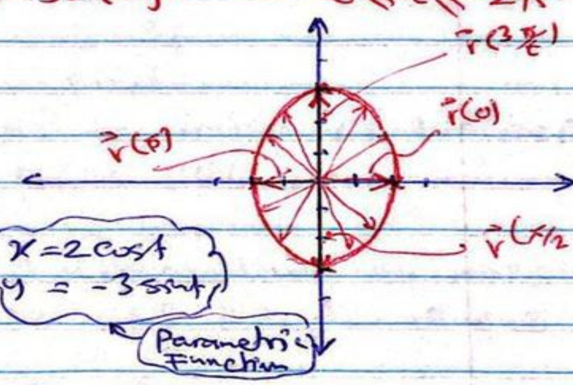


Ex) Graph the vector function

$$\vec{r}(t) = 2\cos t\, i - 3\sin t\, j \qquad 0 \leq t \leq 2\pi$$

Solution

t	x	y
0	2	0
$\pi/2$	0	-3
π	-2	0
$3\pi/2$	0	3
2π	2	0

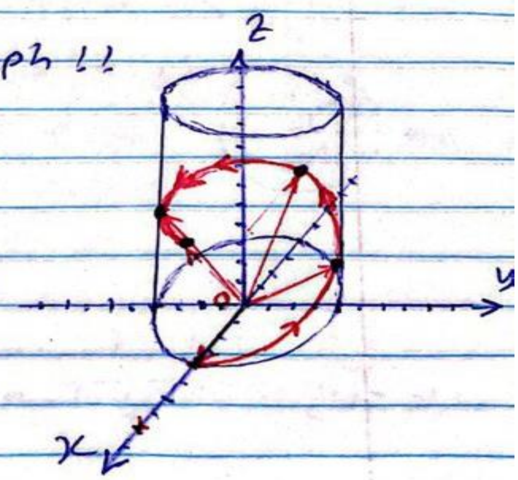


Ex) Graph the vector fun $\vec{r}(t) = 4\cos t\, i + 4\sin t\, j + t\, k$

Solution

Here we have a 3-D graph !!

t	x	y	z
0	4	0	0
$\pi/2$	0	4	$\pi/2 = 1.57$
π	-4	0	$\pi = 3.14$
$3\pi/2$	0	-4	$3\pi/2 = 4.7124$
2π	4	0	$2\pi = 6.28$



Ex) Find a vector & parametric equations for the line segment that joins $A(1,-3,4)$ to $B(-5,1,7)$?

Sol.)

We had solve same of this example past two lectures.

$\vec{r}(t) = \vec{AB} = (-5-1)i + (1-(-3))j + (7-4)k$

$\boxed{\vec{r}(t) = \vec{AB} = \underset{A}{-6}\,i + \underset{B}{4}\,j + \underset{C}{3}\,k}$ **Ans**

let A = initial point at $t=0$ $= (x_0, y_0, z_0) = (1,-3,4)$

$\therefore x = At + x_0 = -6t + 1 = 1 - 6t$

$y = Bt + y_0 = 4t - 3 = -3 + 4t$

$z = Ct + z_0 = 3t + 4 = 4 + 3t$

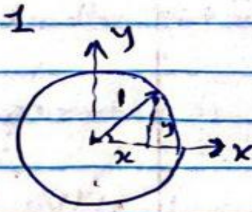
we can also use B as an initial point & reproduce the parametric eq.s $x(t), y(t), z(t)$

Ex) Find a vector Function that represents the curve of intersection of $x^2+y^2=1$ & $y+z=2$?

Solution)

$x^2+y^2=1 \longrightarrow$ circle with radius $=1$

$\therefore \boxed{x=\cos t}$; $\boxed{y=\sin t}$ $\quad 0 \le t \le 2\pi$



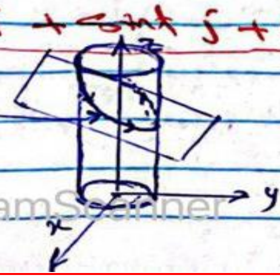
Yet, we have expression for x & y, we need that for z,

$y+z=2 \longrightarrow z = 2-y$

$\therefore \boxed{z = 2-\sin t}$

$\therefore \boxed{\vec{r}(t) = \cos t\, i + \sin t\, j + (2-\sin t)\, j}$ **Ans**

مطلوب: تقاطع الدائرة مع المستوي



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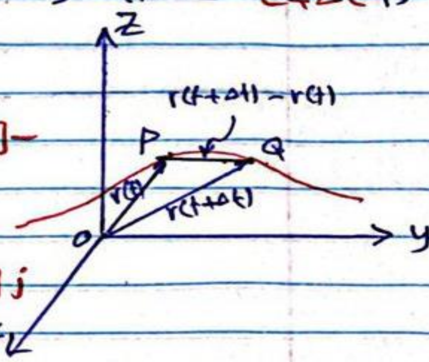
Derivation of Vector valued Functions

If the position vector of a particle moving along a curve in space is;

$\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$

and all of $f(t)$, $g(t)$ & $h(t)$ are differentiable functions of t. Then the difference between the particle's positions at time t & time $t+\Delta t$ is



$$\Delta r = \bar{r}(t+\Delta t) - \bar{r}(t)$$

$$= [f(t+\Delta t)\,i + g(t+\Delta t)\,j + h(t+\Delta t)\,k] - [f(t)\,i + g(t)\,j + h(t)\,k]$$

$$= [f(t+\Delta t) - f(t)]\,i + [g(t+\Delta t) - g(t)]\,j + [h(t+\Delta t) - h(t)]\,k$$

As Δt approaches zero, three things seem to happen simultaneously. 1st, Q approaches P along the curve.
2nd, the secant line PQ seems to approach a limiting position tangent to the curve at P.
3rd, the quotient $\Delta r/\Delta t$ approaches the limit

$$\lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\right] i + \left[\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}\right] j + \left[\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}\right] k$$

$$r'(t) = \frac{dr}{dt} = \left[\frac{df}{dt}\right] i + \left[\frac{dg}{dt}\right] j + \left[\frac{dh}{dt}\right] k$$

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Ex/ Find the velocity, speed & acceleration of a partical whose motion in space is given by the position vector $r(t) = 2\cos t\, i + 2\sin t\, j + 5\cos^2 t\, k$.

Sol.)

The velocity & acceleration vectors at time t are

$$v(t) = r'(t) = \frac{dr(t)}{dt} = -2\sin t\, i + 2\cos t\, j - 10\cos t \sin t\, K$$

$$= \boxed{-2\sin t\, i + 2\cos t\, j - 5\sin 2t\, K} \quad \text{Ans}$$

$$a(t) = v'(t) = \frac{dv(t)}{dt} = \frac{d^2 r(t)}{dt^2} = \boxed{-2\cos t\, i - 2\sin t\, j - 10\cos 2t\, K} \quad \text{Ans}$$

$$\text{speed} = |v(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$$

$$= \boxed{\sqrt{4 + 25\sin^2 2t}}, \quad \text{Ans}$$

Differentiation Rules for Vector Functions:

Let u & v be differentiable vector fns of t, C a constant vector, c any scalar, & f any differentiable scalar fns

① Constant Function Rule: $\rightarrow \frac{d}{dt} C = \text{zero}$

② Scalar Multiple Rules: $\rightarrow \frac{d}{dt}[c\, u(t)] = c\, u'(t)$

$\rightarrow \frac{d}{dt}[f(t)\, u(t)] = f(t)\, u'(t) + u(t)\, f'(t)$

③ Sum Rule: $\rightarrow \frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$

④ Different Rule: $\rightarrow \frac{d}{dt}[u(t) - v(t)] = u'(t) - v'(t)$

⑤ Dot product: $\rightarrow \frac{d}{dt}[u(t) \cdot v(t)] = u(t) \cdot v'(t) + v(t) \cdot u'(t)$

⑥ Cross Product Rule: $\rightarrow \frac{d}{dt}[u(t) \times v(t)] = u(t) \times v'(t) + v(t) \times u'(t)$

⑦ Chain Rule: $\rightarrow \frac{d}{dt}[u(f(t))] = u'(f(t))\, f'(t)$

Ex) proof of the Dot product Rule :-

Solution)

let $u = u_1(t)\, i + u_2(t)\, j + u_3(t)\, k$

$v = v_1(t)\, i + v_2(t)\, j + v_3(t)\, k$

Then,

$$\frac{d}{dt}(u \cdot v) = \frac{d}{dt}(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

$$= \frac{d}{dt}(u_1 v_1) + \frac{d}{dt}(u_2 v_2) + \frac{d}{dt}(u_3 v_3)$$

$$= u_1 v_1' + v_1 u_1' + u_2 v_2' + v_2 u_2' + u_3 v_3' + v_3 u_3'$$

$$= \underbrace{u_1 v_1' + u_2 v_2' + u_3 v_3'}_{u \cdot v'} + \underbrace{v_1 u_1' + v_2 u_2' + v_3 u_3'}_{v \cdot u'}$$

$$\boxed{\frac{d}{dt}(u \cdot v) = u \cdot v' + v \cdot u'} \qquad \text{Ans}$$

Ex) proof of the cross product Rule :-

Solution)

According to the definition of derivative

$$\frac{d}{dt}(u \times v) = \lim_{h \to 0} \frac{u(t+\Delta t) \times v(t+\Delta t) - u(t) \times v(t)}{h}$$

To change this fraction into an equivalent one that contains the difference quotients for the derivative of u & v, we add & subtract "$u(t) \times v(t+h)$" in the numerator, yields:

هنا نقوم اضافة وطرح الحد التالي "$u(t) \times v(t+h)$"

$$\frac{d}{dt}(u \times v) = \lim_{h \to 0} \frac{u(t+h) \times v(t+h) + u(t) \times v(t+h) - u(t) \times v(t+h) - u(t) \times v(t)}{h}$$

$$= \lim_{h \to 0} \left[\frac{u(t+h) - u(t)}{h} \times v(t+h) + u(t) \times \frac{v(t+h) - v(t)}{h} \right]$$

$$= \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \times \lim_{h \to 0} v(t+h) + \lim_{h \to 0} u(t) \times \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

$$= \frac{du}{dh} \times v + u \times \frac{dv}{dt}$$

or

$$= \boxed{u' \times v + u \times v'} = \boxed{u(t) \times v'(t) + v(t) \times u'(t)}$$

Ans

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Maxima & Minima of Two Variables Functions

القيم العظمى والصغرى لدالة ذات متغيرين

Theorem: نظرية

If $f(x,y)$ is a function of two independent variables (x,y) and its 1st & 2nd partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x = f_y = 0$ at point (a,b), then,

① f has a local maximum at (a,b) if $f_{xx} < 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

② f has a local minimum at (a,b) if $f_{xx} > 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

③ f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

غير حاسم / غير مؤكد

④ The test is inconclusive/doubtful at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b). Need to find some other way !!

Notes

The expression $f_{xx} f_{yy} - f_{xy}^2$ is called the discriminant (مميز) of f. To remember it, use,

$$f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

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Ex) Find the local extreme values of

$f(x,y) = x^2 + y^2$

Solution

$f_x = 2x = 0 \longrightarrow \boxed{x=0} = a$

$f_y = 2y = 0 \longrightarrow \boxed{y=0} = b$

$f_{xx} = 2 > 0$

$f_{yy} = 2$

$f_{xy} = 0 \Longrightarrow f_{xy}^2 = 0$

$\therefore f_{xx} f_{yy} - f_{xy}^2 = 2 \times 2 - 0 = 4 > 0$

$\because f_{xx} > 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$, then the point $(0,0)$ is critical point, & the function $f(x,y) = x^2 + y^2$ has local minimum at $(0,0)$.

Ex) Find the extreme value of the function

$F(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$?

Solution

$f_x = y - 2x - 2 \longrightarrow f_{xx} = -2 < 0$

$f_x = 0 \longrightarrow y - 2x - 2 = 0$ —— ①

$f_y = x - 2y - 2 \longrightarrow f_{yy} = -2$

$f_y = 0 \Longrightarrow -2y + x - 2 = 0$ —— ②

حل المعادلتين ① و ② آنياً لإيجاد (x,y)

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$y - 2x - 2 = 0 \quad \text{---} (1)$

بالجمع $\quad -2y + x - 2 = 0 \quad \text{---} (2) \quad \times 2$

$-3y - 6 = 0 \Longrightarrow y = -\frac{6}{3} = \boxed{-2} = b$

نعوض عن قيمة y في اي معادلة (1) أو (2) لإيجاد قيمة x -

$-2 - 2x - 2 = 0 \Longrightarrow -4 - 2x = 0 \Longrightarrow x = \boxed{-2} = a$

∴ الدالة تمتلك حالة متطرفة عند هذه النقطة $(a, b) = (-2, -2)$ ∴

ولمعرفة هذه الحالة نقل الفحص التالي :-

$\boxed{f_{xy} = 1}$

$f_{xx} f_{yy} - f_{xy}^2 = -2 \times -2 - 1^2 = 4 - 1 = 3 > 0$

∴ As $f_{xx} < 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$, then the function $f(x,y)$ has a local maximum at $(-2,-2)$.

$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2 \times (-2) - 2(-2) + 4$

$= \boxed{8}$

Functions of 2 and More Variables, Dept and Indept Variables, Partial Derivatives, PD with Chain Rule, Vector Valued Differentiation, Maxima & Minima Values for 2 Var. Functions -- نهاية محاضرة " دوال بمتغيرين وأكثر، متغيرات معتمدة وغير معتمدة، المشتقات الجزئية، المشتقات الجزئية وقاعدة السلسلة، تفاضل القيم الاتجاهية، النهايات العظمى والصغرى للدوال بمتغيرين"--

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