## 2.4- Non-Exact First Order Differential Equation

The equation which isn't exact can be made exact by multiplying it by the Integrating Factor (I.F).

ان المعادلة الغير تامة **(Non-Exact)** نستطيع جعلها تامة بواسطة ضربها بمعامل التكامل، هو معامل يضرب به المعادلة الغير تامة فيحولها الى معادلة تامة.

**For Example:**

$2xy^3\,dx + 3x^2y^2\,dy = 0$

$M = 2xy^3\,,\;\; N = 3x^2y^2$

$$\left.\frac{\partial M}{\partial y} = 6xy^2, \quad \frac{\partial N}{\partial x} = 6xy^2\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore Exact$$

$if\;\; 2xy^3\,dx + 3x^2y^2\,dy = 0\;\;] \div x$

$2y^3\,dx + 3xy^2\,dy = 0$

$$\left.\frac{\partial M}{\partial y} = 6y^2, \quad \frac{\partial N}{\partial x} = 3y^2\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore Non - \;Exact$$

لو كانت العملية بالعكس فان معامل التكامل هو $(x)$ عند ضرب المعادلة غير التامة يعمل على تحويلها الى معادلة تامة.

$\therefore\, x$ is Integrating Factor

$if\;\; 2xy^3\,dx + 3x^2y^2\,dy = 0\;\;] \div y^2$

$2xy\,dx + 3x^2\,dy = 0$

$$\left.\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 6x\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore Non - \;Exact$$

$\therefore\, y^2$ is integrating factor

**How to Find Integrating Factor:**

If the integrating factor is a function of $(x)$ only $(r)$;

$$M(x,y)\cdot dx + N(x,y)\cdot dy = 0\} \quad *r$$

$$r\,M(x,y)\cdot dx + r\,N(x,y)\cdot dy = 0$$

$$\frac{\partial(rM)}{\partial y} = \frac{\partial(rN)}{\partial x}$$

$$r\frac{\partial M}{\partial y} = r\frac{\partial N}{\partial x} + N\frac{dr}{dx} \quad \rightarrow \quad re-arrangement$$

$$r\frac{\partial M}{\partial y} - r\frac{\partial N}{\partial x} = N\frac{dr}{dx}$$

$$r\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{dr}{dx}$$

$$\frac{dr}{r} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx \quad \rightarrow \quad \int\frac{dr}{r} = \int\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$

$$\ln r = \int\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$ نأخذ $e$ للطرفين

$$\therefore \boldsymbol{r_x = e^{\int\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx}}$$ integrating factar is a function of $(x)$

by the same method get to:

$$\boldsymbol{r_y = e^{\int\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy}}$$ integrating factar is a function of $(y)$

**Example (1): Find a general solution for $\boldsymbol{y\cdot dx + (3+3x-y)\cdot dy = 0}$?**

**Solve:**

$$M = y,\quad N = (3+3x-y)$$

$$\left.\frac{\partial M}{\partial y} = 1,\quad \frac{\partial N}{\partial x} = 3\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore Non-Exact$$

$\therefore$ Find integrating factor:

$$\therefore r_x = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx} = e^{\int \frac{1}{(3+3x-y)}(1-3) dx} = e^{\int \frac{-2}{(3+3x-y)} \times \frac{3}{3} dx}$$

$$= e^{-\frac{2}{3}\ln(3+3x-y)} = e^{\ln(3+3x-y)^{-2/3}} = (3+3x-y)^{-2/3}$$

يهمل هذا المعامل لأنه دالة من $x$ و $y$.

$$r_y = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy} = e^{\int \frac{1}{y}(3-1) dy} = e^{\int \frac{2}{y} dy} = e^{2\ln y} = e^{\ln y^2}$$

$$r_y = y^2$$

$\therefore y^2$ is integrating factor

$$y \cdot dx + (3+3x-y) \cdot dy = 0\} * y^2$$

$$y^3 \cdot dx + (3y^2 + 3xy^2 - y^3) \cdot dy = 0$$

$$M = y^3, \quad N = (3y^2 + 3xy^2 - y^3)$$

$$\left.\frac{\partial M}{\partial y} = 3y^2, \quad \frac{\partial N}{\partial x} = 3y^2\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore Exact$$

To find $f(x,y)$:

$$M = \frac{df}{dx} = y^3 \quad \rightarrow \quad \int df = \int y^3 \cdot dx$$

$$\therefore f(x,y) = xy^3 + g(y)$$

$$\frac{df}{dy} = 3xy^2 + \frac{dg}{dy} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

$$\because N = \frac{df}{dy} = 3y^2 + 3xy^2 - y^3 \dots\dots\dots\dots\dots (2)$$

Sub eq. (1) and eq. (2) to get:

$$3xy^2 + \frac{dg}{dy} = 3y^2 + 3xy^2 - y^3 \quad \rightarrow$$

$$\frac{dg}{dy} = 3y^2 - y^3 \quad \rightarrow \quad \int dg = \int (3y^2 - y^3) \cdot dy + c$$

$$g(y) = y^3 - \frac{y^4}{4} + c$$

$$\therefore f(x,y) = xy^3 + y^3 - \frac{y^4}{4} + c$$

**Example (2):** **Show that $\frac{1}{x^2+y^2}$ is an integrating factor for the equation $(x^2+y^2-x)\cdot dx - y\cdot dy = 0$ ?**

**Solve:**

$M = (x^2+y^2-x), \quad N = -y$

$$\left.\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 0\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore Non-\ Exact$$

$$(x^2+y^2-x)\cdot dx - y\cdot dy = 0\} \ * \ \frac{1}{x^2+y^2}$$

$$\left(1-\frac{x}{x^2+y^2}\right)\cdot dx - \frac{y}{x^2+y^2}\cdot dy = 0$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y} = \left(0-\frac{0-x\cdot 2y}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2} \\ \frac{\partial N}{\partial x} = \left(-\frac{0-y\cdot 2x}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2}\end{array}\right\} \quad \rightarrow \quad \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore Exact$$

$\therefore \frac{1}{x^2+y^2}$ is integrating factor

## Problems:

**H.W-** Check the given expression is **Exact** or **Non-Exact** differential equation. if the expression is **Non-Exact** equation, Find the Integration Factor (**I.F**) for:

1) $(x-y)\cdot dx - dy = 0$

**Answer**: Non-Exact, I.F: $e^x$

2) $(y^2-1)\cdot dx + (2xy-\sin y)\cdot dy = 0$

**Answer**: Exact

3) $y\cdot dx + (4x-y^2)\cdot dy = 0$

**Answer**: Non-Exact, I.F: $y^3$

4) $(y^2-x^2)\cdot dy - 2xy\cdot dx = 0$

**Answer**: Exact

5) $(2y\sin x - 3)\cdot dx - \cos x\cdot dy = 0$

**Answer**: Non-Exact, I.F: $\cos x$