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Lecture#4

Friday, November 5, 2021 11:22 AM

## 2.2 MATHEMATICAL TOOLS FOR MODELING

As we just saw in our analysis of the chemical mixer, the unsteady-state material, and energy balance models that we wrote required us to solve differential equations to obtain the concentration and temperature versus time behavior for the process. This will be a common occurrence for us as we continue our studies of process dynamics and control. It would be beneficial to review some additional tools available to us for solving our process models. In Sec. 2.1, we solved the equations by separation and integration. A couple of other useful tools for solving such models are Laplace transforms and MATLAB/Simulink. In the next several sections, we will review the use of these additional tools for solving our model differential equations.

**Definition of the Laplace Transform**

The Laplace transform of a function $f(t)$ is *defined* to be $F(s)$ according to the equation

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

Exampl 2.1: Find the Laplace transform of function $f(t) = 1$

Solution: Since

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Therefore

$$F(s) = \int_0^\infty (1)\, e^{-st}\, dt$$

$$F(s) = \int_0^\infty e^{-st}\, dt$$

since $\int e^x\, dx = e^x$

$-st \Rightarrow -s\, dt$

$$F(s) = -\frac{1}{s} \int_0^\infty e^{-st} \cdot -s\, dt$$

$$F(s) = -\frac{1}{s} \Big[ e^{-st} \Big]_0^\infty$$

$$F(s) = -\frac{1}{s} \left[ e^{-s(\infty)} - e^{-s(0)} \right]$$

$$F(s) = -\frac{1}{s} \left[ e^{-\infty} - e^{0} \right]$$

$$F(s) = -\frac{1}{s} \left[ \frac{1}{e^{\infty}} - e^{0} \right]$$

since $e^0 = 1$

$\frac{1}{\infty} = 0$

$$F(s) = -\frac{1}{s} [0 - 1]$$

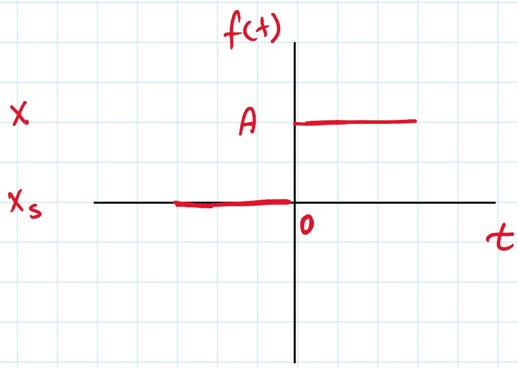
$$F(s) = \frac{1}{s}$$

| Function, f(t) | Laplace Transform, F(s) | Example |
|---|---|---|
| **1** | $\frac{1}{s}$ | $f(t) = 1 \rightarrow F(s) = \frac{1}{s}$ |
| $\boldsymbol{a}$ | $\frac{a}{s}$ | $f(t) = 4 \rightarrow F(s) = \frac{4}{s}$ |
| $\boldsymbol{t}$ | $\frac{1}{s^2}$ | $f(t) = 4t \rightarrow F(s) = \frac{4}{s^2}$ |
| $\boldsymbol{t^n}$ | $\frac{n!}{s^{n+1}}$ | $f(t) = t^4 \rightarrow F(s) = \frac{24}{s^5}$ |
| $\sin \boldsymbol{at}$ | $\frac{a}{s^2 + a^2}$ | $f(t) = \sin 4t \rightarrow F(s) = \frac{4}{s^2 + 16}$ |
| $\cos \boldsymbol{at}$ | $\frac{s}{s^2 + a^2}$ | $f(t) = \cos 4t \rightarrow F(s) = \frac{s}{s^2 + 16}$ |
| $\sinh \boldsymbol{at}$ | $\frac{a}{s^2 - a^2}$ | $f(t) = \sinh 4t \rightarrow F(s) = \frac{4}{s^2 - 16}$ |
| $\cosh \boldsymbol{at}$ | $\frac{s}{s^2 - a^2}$ | $f(t) = \cosh 4t \rightarrow F(s) = \frac{s}{s^2 - 16}$ |
| $\boldsymbol{e^{-at}}$ | $\frac{1}{s+a}$ | $f(t) = e^{-4t} \rightarrow F(s) = \frac{1}{s+4}$ |
| $\boldsymbol{e^{at}}$ | $\frac{1}{s-a}$ | $f(t) = e^{4t} \rightarrow F(s) = \frac{1}{s-4}$ |
| $\boldsymbol{te^{-at}}$ | $\frac{1}{(s+a)^2}$ | $f(t) = te^{-4t} \rightarrow F(s) = \frac{1}{(s+4)^2}$ |
| $\boldsymbol{t^n e^{-at}}$ | $\frac{n!}{(s+a)^{n+1}}$ | $f(t) = t^4 e^{-4t} \rightarrow F(s) = \frac{24}{(s+4)^5}$ |
| $\boldsymbol{e^{-at}} \sin \boldsymbol{wt}$ | $\frac{w}{(s+a)^2 + w^2}$ | $f(t) = e^{-4t} \sin 3t \rightarrow F(s) = \frac{3}{(s+4)^2 + 9}$ |
| $\boldsymbol{e^{-at}} \cos \boldsymbol{wt}$ | $\frac{s+a}{(s+a)^2 + w^2}$ | $f(t) = e^{-4t} \cos 3t \rightarrow F(s) = \frac{s+4}{(s+4)^2 + 9}$ |

**Transforms of Simple Functions**

1- Step function:-

$$f(t) = \begin{bmatrix} 0 & t < 0 \\ A & t > 0 \end{bmatrix}$$




$$f(t) = 0 + A$$

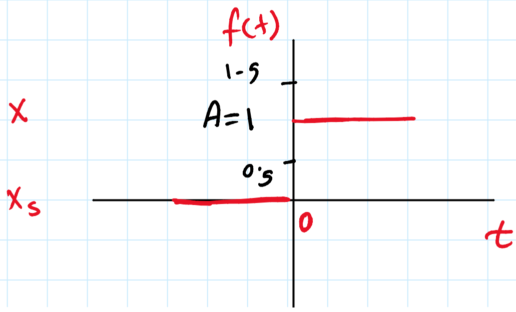
$$F(s) = \frac{A}{S}$$

where $A = x(t) - x_s$ : magnitude of step change

unit step chang, u(t)

$$A = 1$$

$$f(t) = \begin{bmatrix} 0 & t < 0 \\ 1 & t > 0 \end{bmatrix}$$



$$1 \quad t>0 \}$$


$$f(t) = 0 + 1$$

since unit step chang $u(t)$

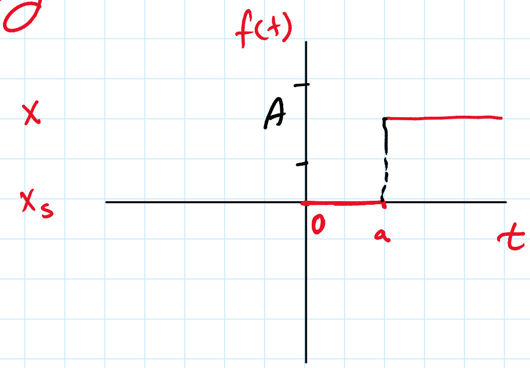
$$f(t) = 0 + u(t)$$

$$L\{u(t)\} = \frac{1}{s}$$

$$F(s) = \frac{1}{s}$$

Step function with time delay

$$f(t) = \begin{bmatrix} 0 & t<a \\ A & t>a \end{bmatrix}$$



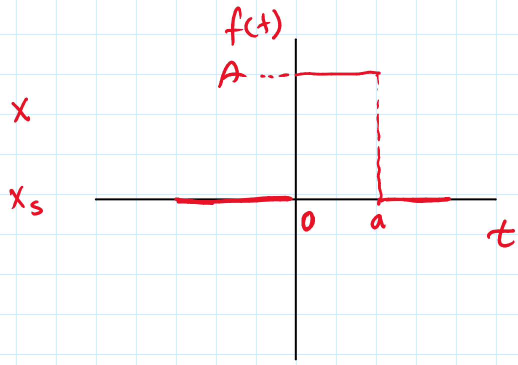

$$f(t) = 0 + A$$

$$f(t) = A$$

$$F(s) = \frac{A}{s} e^{-as}$$

2- pulse function

$$f(t) = \begin{bmatrix} 0 & t<0 \\ A & 0<t<a \\ 0 & t>0 \end{bmatrix}$$




$$f(t) = 0 + Au(t) - Au(t-a)$$

$$f(t) = Au(t) - Au(t-a)$$

$$\boxed{F(s) = \frac{A}{s} - \frac{A}{s} e^{-as}}$$

from table directly

another way to find Laplace transform for pulse functio

## another way to find Laplace transform for pulse function

since

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Therefore

$$F(s) = \underbrace{\int_{-\infty}^{0} 0\, e^{-st}\, dt}_{0} + \int_0^a A\, e^{-st}\, dt + \underbrace{\int_0^\infty 0\, e^{-st}\, dt}_{0}$$

$$F(s) = \int_0^a A\, e^{-st}\, dt \qquad -st \Rightarrow -s\,dt$$

$$F(s) = -\frac{1}{s}\int_0^a A\, e^{-st} * -s\,dt$$

$$F(s) = -\frac{A}{s}\left[e^{-st}\right]_0^a$$

$$F(s) = -\frac{A}{s}\left[e^{-as} - e^{0}\right]$$

$$F(s) = -\frac{A}{s}\left[e^{-as} - 1\right]$$

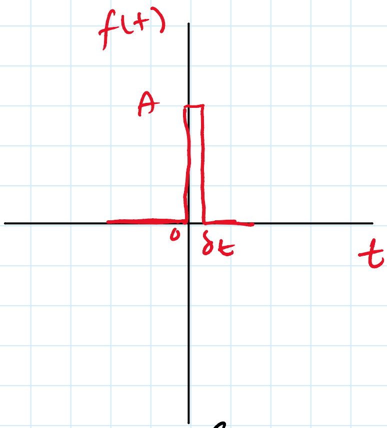
$$F(s) = -\frac{A}{s}e^{-as} + \frac{A}{s}$$

$$\boxed{F(s) = \frac{A}{s} - \frac{A}{s}e^{-as}}$$

Note : If $A = 1 \Rightarrow$ unit pulse

## 3- Impulse function

$$f(t) = \begin{bmatrix} 0 & t < 0 \\ A & 0 < t < \delta_t \\ 0 & t > \delta_t \end{bmatrix}$$




$$F(s) = \text{Area} = A * \delta_t$$

if $A = 1$ = unit impulse

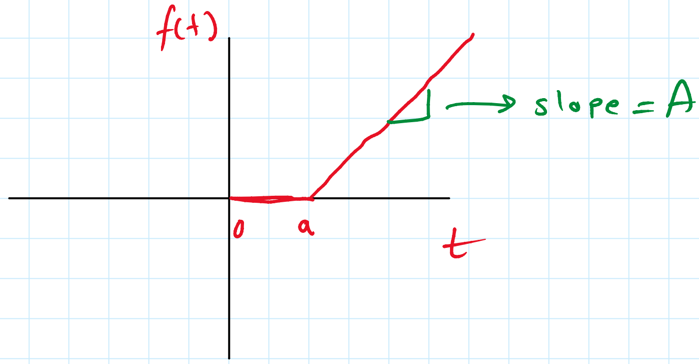
$\delta_t$ is very small

$\delta_t <$ 5% steady state time

## 4. Ramp function

## 4. Ramp function

$$f(t)=\begin{bmatrix} 0 & t<a \\ At & t>a \end{bmatrix}$$




$$f(t) = 0 + At\,u(t-a)$$

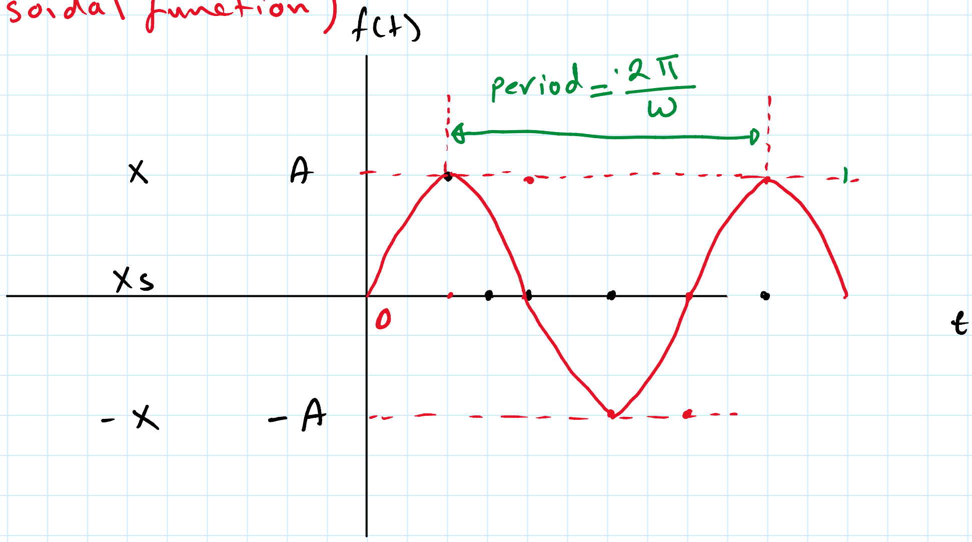
$$f(t) = At\,u(t-a)$$

$$F(s) = A\cdot\left(\frac{1}{s^2}\right)e^{-as}$$

$$F(s) = \frac{A}{s^2}\,e^{-as}$$

## 5. Sine function (Sinusoidal function)

$$f(t)=\begin{bmatrix} 0 & t<0 \\ A\sin\omega t & t>0 \end{bmatrix}$$

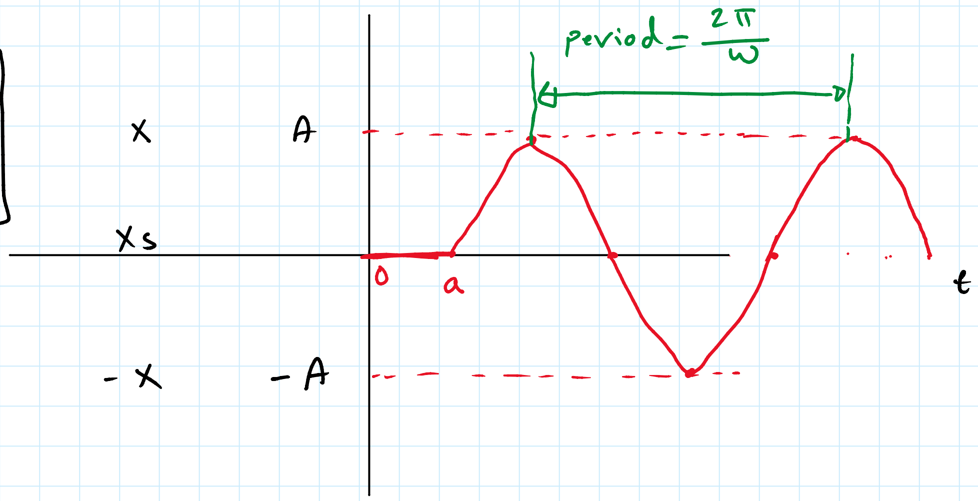



$$F(t) = 0 + A\sin\omega t$$

$$F(s) = \frac{A\ \omega}{s^2+\omega^2}$$

If

$$f(t)=\begin{bmatrix} 0 & t<a \\ A\sin\omega t & t>a \end{bmatrix}$$




$$F(s) = \frac{A\omega}{s^2+\omega^2}\,e^{-as}$$