Email: abbas.jawad@uomus.edu.iq

Lecture#5

Friday, November 12, 2021 12:36 PM

Mathematical Representation of input signal

Example 1: Determine f(t) and F(s) for input signal in the below figure

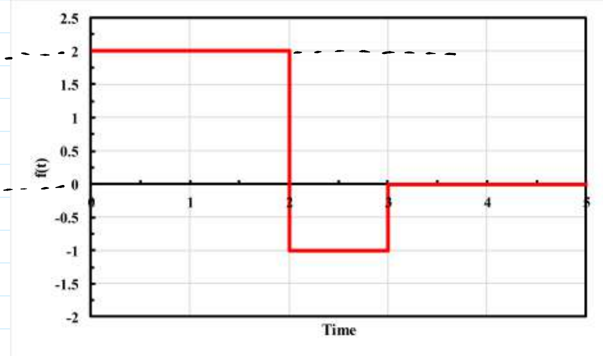
Solution:-

$$f(t) = \begin{bmatrix} 2 & 0 \leq t < 2 \\ -3 & 2 \leq t < 3 \\ 1 & t \geq 3 \end{bmatrix}$$

x

$A = x - x_s$
$= 2 - 0 = 2$

$x_s = 0$



$$f(t) = 2\,u(t-0) - 3\,u(t-2) + (1)u(t-3)$$

$$f(t) = 2\,u(t) - 3\,u(t-2) + u(t-3)$$

$A = x - x_s$
$= 2 - (-1) = 3$

$A = 0 - (-1) = 1$

By taking Laplace Transform for the above eq.

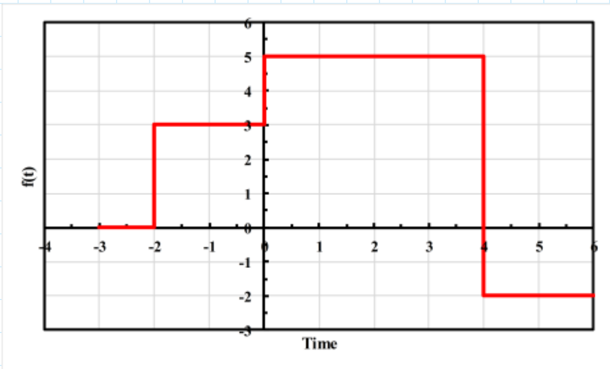
$$F(s) = \frac{2}{s} - \frac{3}{s}e^{-2s} + \frac{1}{s}e^{-3s}$$

Note: upward switching $\longrightarrow$ +ve
downward switching $\longrightarrow$ -ve

Example 2: Find f(t) & F(s) for the following figure.

Solution:—

$$f(t) = \begin{bmatrix} 0 & t \leq -2 \\ 3 & -2 \leq t < 0 \\ 2 & 0 \leq t < 4 \\ -7 & t \geq 4 \end{bmatrix}$$



$$f(t) = \cancel{0\,u(t)}^{\,0} + 3\,u(t+2) + 2\,u(t-0) - 7\,u(t-4)$$

$A = x - x_s$
$= -2 - (5) = -7$

$$f(t) = 3\,u(t+2) + 2\,u(t) - 7\,u(t-4)$$

By taking Laplace Transform for the above equation

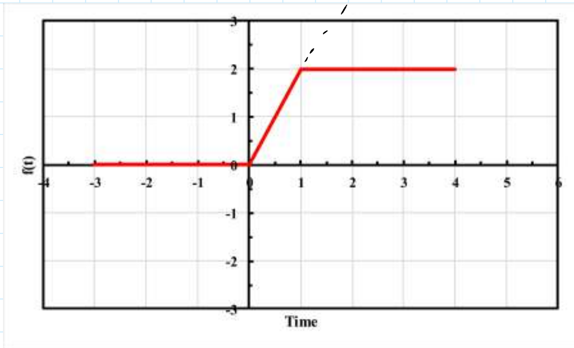
$$F(s) = \frac{3}{s}e^{2s} + \frac{2}{s} - \frac{7}{s}e^{-4s}$$

Example 3:- Find f(t) & F(s) for the following figure.

Solution:-

Example 3:- Find $f(t)$ & $F(S)$ for the following figure.

Solution:-



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$$

$$f(t) = 0 + 2r(t-0) - 2r(t-1)$$

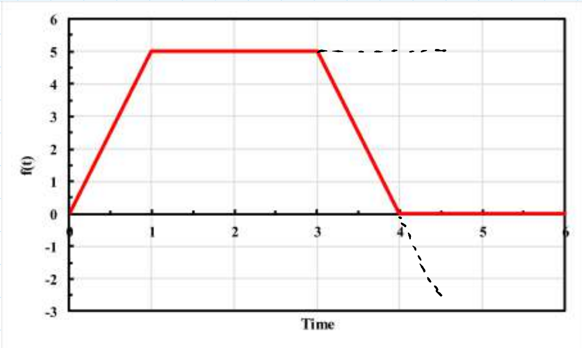
$$f(t) = 2r(t) - 2r(t-1)$$

By taking Laplace Transform

$$F(s) = \frac{2}{s^2} - \frac{2}{s^2} e^{-s}$$

Example 4:- Find $f(t)$ & $F(s)$ for the following figure

Solution:-



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{5}{1} = 5$$

$$f(t) = 5r(t-0) - 5r(t-1) - 5r(t-3) + 5r(t-4)$$

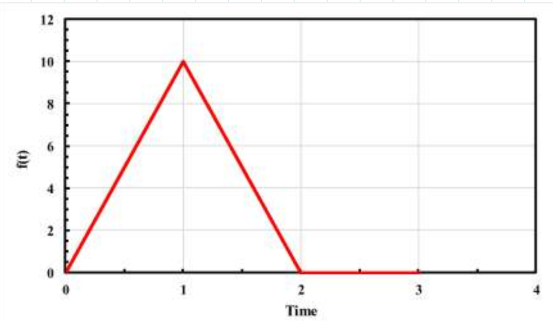
$$f(t) = 5r(t) - 5r(t-1) - 5r(t-3) + 5r(t-4)$$

By taking Laplace Transform for both sides

$$F(s) = \frac{5}{s^2} - \frac{5}{s^2} e^{-s} - \frac{5}{s^2} e^{-3s} + \frac{5}{s^2} e^{-4s}$$

Example 5: Find $f(t)$ & $F(s)$ for the following figure

Solution:-

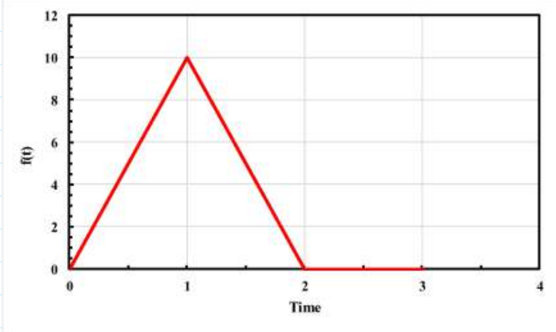


$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{10}{1} = 10$$

$$f(t) = 10r(t-0) - 20r(t-1) + 10r(t-2)$$

$$f(t) = 10r(t) - 20r(t-1) + 10r(t-2)$$

Solution:-



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{10}{1} = 10$$

$$f(t) = 10\,r(t-0) - 20\,r(t-1) + 10\,r(t-2)$$

$$f(t) = 10\,r(t) - 20\,r(t-1) + 10\,r(t-2)$$

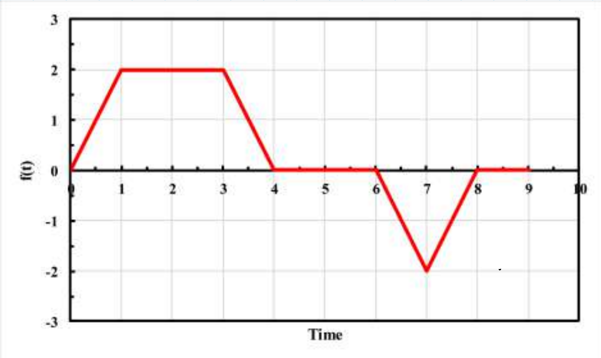
By taking Laplace Transform

$$F(s) = \frac{10}{s^2} - \frac{20}{s^2}e^{-s} + \frac{10}{s^2}e^{-2s}$$

$$F(s) = \frac{10}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right]$$

Example 6: Find f(t) & F(s) for the following figure.

Solution:-



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$$

$$f(t) = 2\,r(t-0) - 2r(t-1) - 2r(t-3) + 2r(t-4) - 2r(t-6) + 4r(t-7) - 2r(t-8)$$

$$f(t) = 2\,r(t) - 2r(t-1) - 2r(t-3) + 2r(t-4) - 2r(t-6) + 4r(t-7) - 2r(t-8)$$

By taking Laplace transform

$$F(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-s} - \frac{2}{s^2}e^{-3s} + \frac{2}{s^2}e^{-4s} - \frac{2}{s^2}e^{-6s} + \frac{4}{s^2}e^{-7s} - \frac{2}{s^2}e^{-8s}$$

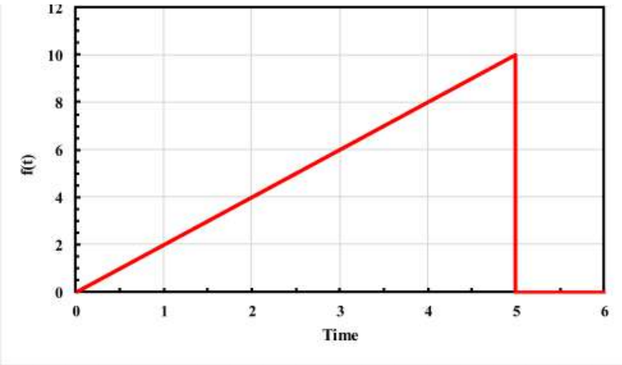
$$F(s) = \frac{2}{s^2}\left[1 - e^{-s} - e^{-3s} + e^{-4s} - e^{-6s} + 2e^{-7s} - e^{-8s}\right]$$

Example 7:- Find f(t) & F(s) for the following figure

Solution:-

$$\text{slope} = \frac{\Delta y}{} = \frac{\text{rise}}{} = \frac{\overset{2}{\cancel{10}}}{} = 2$$

$$slope = \frac{\Delta y}{\Delta x} = \frac{rise}{run} = \frac{\cancel{10}^{2}}{\cancel{5}_{1}} = 2$$



$$f(t) = 2\,r(t-0) - 2\,r(t-5) - 10\,u(t-5)$$

$$f(t) = 2r(t) - 2r(t-5) - 10\,u(t-5)$$

By taking Laplace Transform

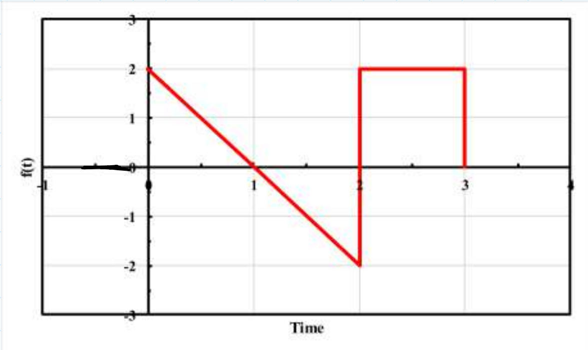
$$F(s) = \frac{2}{s^2} - \frac{2}{s^2} e^{-5s} - \frac{10}{s} e^{-5s}$$

$$F(s) = 2\left[\frac{1}{s^2} - \frac{1}{s^2} e^{-5s} - \frac{5}{s} e^{-5s}\right]$$

Example 8: Find f(t) & F(s) for the following figure

Solution:-

$$slope = \frac{\Delta y}{\Delta x} = \frac{rise}{run} = \frac{2}{1} = 2$$



$$f(t) = 0 + 2\,u(t-0) - 2\,r(t-0) + 2r(t-2) + 4\,u(t-2) - 2\,u(t-3)$$

$$f(t) = 2\,u(t) - 2r(t) + 2r(t-2) + 4\,u(t-2) - 2u(t-3)$$

By taking Laplace transform

$$F(s) = \frac{2}{s} - \frac{2}{s^2} + \frac{2}{s^2} e^{-2s} + \frac{4}{s} e^{-2s} - \frac{2}{s} e^{-3s}$$

Transform of Derivatives

General case

$$\boxed{L\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots f^{(n-1)}(0)}$$

for example:

$$L\left\{\frac{df}{dt}\right\} = s^1 F(s) - s^{1-1} f(0) - \cancel{s^{1-2} f'(0)} - \cancel{f^{(1-1)}(0)}$$

$$L\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

Example: find Laplace Transform for $\frac{d^2f}{dt^2}$

$$L\left\{\frac{d^2f}{dt^2}\right\} = s^2 F(s) - sf(0) - f'(0)$$

Example: find Laplace Transform for $\frac{d^3f}{dt^3}$

$$L\left\{\frac{d^3f}{dt^3}\right\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

Solution of ordinary differential equations (ODEs) by Laplace Transform

Example: Solve the following equation

$$\frac{dX}{dt} + 3X = 0 \quad , \; X(0) = 2$$

Solution:-

$$\frac{dX}{dt} + 3X = 0 \qquad \text{——} (1)$$

By taking Laplace transform for both sides

$$sX(s) - X(0) + 3X(s) = 0 \qquad \text{——} (2)$$

since $x(0)=2$ sub. into Eq. ②

$$s\,X(s) - 2 + 3\,X(s) = 0$$

$$s\,X(s) + 3\,X(s) = 2$$

$$X(s)(s+3) = 2$$

$$X(s) = \frac{2}{s+3}$$

$$X(s) = 2 \cdot \frac{1}{s+3}$$

By taking Laplace inverse for both sides

$$\mathcal{L}^{-1} X(s) = \mathcal{L}^{-1}\, 2\, \frac{1}{s+3}$$

$$x(t) = 2\, \mathcal{L}^{-1} \frac{1}{s+3} \quad \text{—— ③}$$

since $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

$$\mathcal{L}^{-1} \frac{1}{s+a} = e^{-at}$$

$$\mathcal{L}^{-1} \frac{1}{s+3} = e^{-3t} \qquad \text{sub. into Eq. ③}$$

$$x(t) = 2\, e^{-3t}$$

Example: Solve the following equation by Laplace Transform

$$y'' + 6y' + 9y = 0 \;,\; y(0) = -1 \;,\; y'(0) = 6$$

Solution:

$$y'' + 6y' + 9y = 0$$

By taking Laplace transform for both sides

$$s^2 y(s) - s y(0) - y'(0) + 6[s y(s) - y(0)] + 9 y(s) = 0 \quad \text{—} ①$$

since $y(0) = -1$ & $y'(0) = 6$ sub. into Eq. ①

$$s^2 y(s) - s(-1) - 6 + 6[s y(s) - (-1)] + 9 y(s) = 0$$

$$s^2 y(s) + s \cancel{-6} + 6 s y(s) \cancel{+6} + 9 y(s) = 0$$

$$s^2 y(s) + 6 s y(s) + 9 y(s) = -s$$

$$y(s)(s^2 + 6s + 9) = -s$$

$$y(s) = \frac{-s}{s^2 + 6s + 9}$$

By taking Laplace inverse for both sides

$$\mathcal{L}^{-1} y(s) = \mathcal{L}^{-1} \frac{-s}{s^2 + 6s + 9}$$

$$y(t) = -\mathcal{L}^{-1} \frac{s}{s^2 + 6s + 9} \quad \text{—} ②$$

since $(s+3)^2 = s^2 + 6s + 9$

Therefore

$$y(t) = -\mathcal{L}^{-1} \frac{s}{(s+3)^2}$$

$$y(t) = -\mathcal{L}^{-1} \frac{s + 3 - 3}{(s+3)^2}$$

$$y(t) = -\left[\mathcal{L}^{-1} \frac{\cancel{s+3}}{(s+3)^{\cancel{2}}} - \mathcal{L}^{-1} \frac{3}{(s+3)^2} \right]$$

$$y(t) = -\left[\mathcal{L}^{-1} \frac{1}{\quad} - \mathcal{L}^{-1} \frac{3}{\quad} \right]$$

$$y(t) = -\left[\mathcal{L}^{-1}\frac{1}{s+3} - \mathcal{L}^{-1}\frac{3}{(s+3)^2}\right]$$

$$y(t) = -\left[\mathcal{L}^{-1}\frac{1}{s+3} - 3\,\mathcal{L}^{-1}\frac{1}{(s+3)^2}\right] \quad \text{——— (3)}$$

from the table of Laplace inverse, we obtain

$$\mathcal{L}^{-1}\frac{1}{s+a} = e^{-at}$$

$$\mathcal{L}^{-1}\frac{1}{(s+a)^2} = te^{-at}$$

$$y(t) = -\left[e^{-3t} - 3te^{-3t}\right]$$

$$y(t) = -e^{-3t} + 3te^{-3t}$$

$$y(t) = 3te^{-3t} - e^{-3t}$$

$$y(t) = e^{-3t}(3t-1)$$