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Numerical Analysis

Lecture 11

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الرياضيات المتقدمة: المرحلة الثانية

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1 Least Squares Approximations

Given a set of data points (x_i, y_i) for $i = 1, 2, \ldots, n$, the least squares approximation seeks to find the line $y = mx + b$ that minimizes the sum of the squares of the vertical distances from the points to the line.

The line of best fit is determined by solving the following system of equations for m and b:

$$\sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i$$
$$\sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + nb$$

Solving these equations, we get:

$$m = \frac{n \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$b = \frac{\sum_{i=1}^{n} y_i - m \sum_{i=1}^{n} x_i}{n}$$

Thus, the equation of the line of best fit is:

$$y = mx + b$$

Example 1.1. Find the following points to linear form $y = a + bx$, where

$$(x_1, y_1) = (1, 3)$$
$$(x_2, y_2) = (2, 5)$$
$$(x_3, y_3) = (3, 8)$$
$$(x_4, y_4) = (4, 13)$$
$$(x_5, y_5) = (5, 16)$$

Sol. To find the linear equation in the form $y = a + bx$, we will calculate the slope b and the intercept a.

Step 1: Calculate the Necessary Sums

$$n = 5$$

$$\sum x_i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum y_i = 3 + 5 + 8 + 13 + 16 = 45$$

$$\sum x_i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$\sum x_i y_i = 1 \cdot 3 + 2 \cdot 5 + 3 \cdot 8 + 4 \cdot 13 + 5 \cdot 16 = 3 + 10 + 24 + 52 + 80 = 169$$

Step 2: Calculate b and a

The formulas for the slope b and intercept a are given by:

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i - b \sum x_i}{n}$$

Calculating b:

$$b = \frac{5 \cdot 169 - 15 \cdot 45}{5 \cdot 55 - 15^2}$$

Simplifying:

$$b = \frac{845 - 675}{275 - 225} = \frac{170}{50} = 3.4$$

Calculating a:

$$a = \frac{45 - 3.4 \cdot 15}{5} = \frac{45 - 51}{5} = \frac{-6}{5} = -1.2$$

Final Linear Equation

Thus, the linear equation that best fits the given data points is:

$$y = -1.2 + 3.4x$$

□

Example 1.2. Find the following points to linear form $y = ae^{bx}$, where

$$(x_1, y_1) = (0, 1.5)$$
$$(x_2, y_2) = (1, 2.5)$$
$$(x_3, y_3) = (2, 3.5)$$
$$(x_4, y_4) = (3, 5)$$
$$(x_5, y_5) = (4, 7.5)$$

Sol. To find the exponential equation in the form $y = ae^{bx}$, we will linearize the equation using the natural logarithm.

Step 1: Linearization

Taking the natural logarithm of both sides:

$$\ln(y) = \ln(a) + bx$$

Setting $Y = \ln(y)$, $A = \ln(a)$, $B = b$, and $X = x$.

Step 2: Calculate the Data Points

We are given the following data points:

$$x_1 = 0, \quad y_1 = 1.5 \quad \Rightarrow Y_1 = \ln(1.5) \approx 0.4055$$

$$x_2 = 1, \quad y_2 = 2.5 \quad \Rightarrow Y_2 = \ln(2.5) \approx 0.9163$$

$$x_3 = 2, \quad y_3 = 3.5 \quad \Rightarrow Y_3 = \ln(3.5) \approx 1.2528$$

$$x_4 = 3, \quad y_4 = 5 \quad \Rightarrow Y_4 = \ln(5) \approx 1.6094$$

$$x_5 = 4, \quad y_5 = 7.5 \quad \Rightarrow Y_5 = \ln(7.5) \approx 2.0149$$

Step 3: Calculate b and a

The necessary sums are:

$$n = 5$$

$$\sum x_i = 0 + 1 + 2 + 3 + 4 = 10$$

$$\sum y_i \approx 0.4055 + 0.9163 + 1.2528 + 1.6094 + 2.0149 \approx 6.1989$$

$$\sum x_i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum x_i y_i \approx 0 + 0.9163 + 2.5056 + 4.8282 + 8.0596 \approx 16.3097$$

Using the formulas:

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

Calculating b and A:

$$b = \frac{5 \cdot 16.3097 - 10 \cdot 6.1989}{5 \cdot 30 - 10^2}$$

Simplifying:

$$b = \frac{81.5485 - 61.989}{150 - 100} = \frac{19.5595}{50} \approx 0.3912$$

$$A = \frac{\sum Y_i - b \sum x_i}{n}$$

$$A = \frac{6.1989 - 0.3912 \cdot 10}{5} = \frac{6.1989 - 3.912}{5} = \frac{2.2869}{5} \approx 0.4574$$

Calculating a:

$$a = e^A = e^{0.4574} \approx 1.59$$

Final Linear Equation

Thus, the exponential equation that best fits the given data points is:

$$y = ae^{bx}$$

$$y = 1.59e^{0.3912x}$$

□

Homework of Least Squares Approximations

1. Find the following points to linear form $y = a + bx$, where

$$(x_1, y_1) = (1, 1)$$

$$(x_2, y_2) = (2, 5)$$

$$(x_3, y_3) = (3, 7)$$

$$(x_4, y_4) = (4, 9)$$

$$(x_5, y_5) = (5, 11)$$

2. Find the following points to linear form $y = ae^{bx}$, where

$$(x_1, y_1) = (0, 0.5)$$

$$(x_2, y_2) = (1, 1)$$

$$(x_3, y_3) = (2, 1.5)$$

$$(x_4, y_4) = (3, 2)$$

$$(x_5, y_5) = (4, 2.5)$$