2nd class

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Numerical Analysis

Lecture 12

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الرياضيات المتقدمة: المرحلة الثانية

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1 Introduction to Fourier Series

In mathematics, the Fourier series is a method for expressing a periodic function as a sum of sine and cosine functions. The concept was introduced by Jean-Baptiste Joseph Fourier in the early 19th century, and it has since become a fundamental tool in various fields such as signal processing, physics, and engineering.

Definition 1.1 (Periodic Functions). A function $f(x)$ is called periodic if there exists a positive number T such that:

$$f(x+T) = f(x)$$

for all values of x. The smallest positive value of T is called the fundamental period of the function.

For example:

1. **Sine and Cosine Functions:**

$$\sin(x+2\pi) = \sin(x), \quad \cos(x+2\pi) = \cos(x)$$

The period of both the sine and cosine functions is 2π.

2. **Square Wave:**

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < \pi \\ -1, & \text{if } \pi \leq x < 2\pi \end{cases}$$

This function has a period of 2π.

Definition 1.2 (Fourier series). Given a periodic function $f(x)$ with period 2π, the Fourier series of $f(x)$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the coefficients a_0, a_n, and b_n are defined as:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

Remark 1.1. If the function $f(x)$ defined on interval $-\pi < x < \pi$, then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Example 1.1. Find Fourier series of the function $f(x) = x$, from$x = 0$ to $x = 2\pi$

Sol. To find the Fourier series of $f(x) = x$ defined on the interval $[0, 2\pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Step 1: Calculating a_0

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x\, dx = \frac{1}{2\pi} \left. \frac{x^2}{2} \right|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{(2\pi)^2}{2} = \pi$$

Step 2: Calculate a_n

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \cos(nx)\, dx$$

Using integration **(**$uv - \int vdu$**)** by parts, let $u = x$ and $dv = \cos(nx)\, dx$:

$$du = dx, \quad v = \frac{\sin(nx)}{n}$$

$$a_n = \frac{1}{\pi} \left[\frac{x \sin(nx)}{n} \Big|_0^{2\pi} - \int_0^{2\pi} \frac{\sin(nx)}{n}\, dx \right] = 0$$

since the integral of $\sin(nx)$ over $[0, 2\pi]$ is zero

Step 3: Calculate b_n

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin(nx)\, dx$$

Using integration by parts, let $u = x$ and $dv = \sin(nx)\, dx$:

$$du = dx, \quad v = -\frac{\cos(nx)}{n}$$

$$b_n = \frac{1}{\pi} \left[-\frac{x \cos(nx)}{n} \Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos(nx)}{n}\, dx \right] = \frac{1}{\pi} \left[\frac{2\pi}{n} \right] = -\frac{2}{n}$$

So, the Fourier series for $f(x) = x$ in the interval $[0, 2\pi]$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$f(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$$

$\square$

Homework 1: Fourier Series

Find Fourier series of the function $f(x) = 5x$, from $x = 0$ to $x = 2\pi$