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Numerical Analysis

Lecture 5

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الرياضيات المتقدمة: المرحلة الثانية

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Contents

1 Rung Kutta Method

The Runge-Kutta Method of order 4 (RK4) is one of the most commonly used methods for solving ordinary differential equations (ODEs). It provides a good balance between accuracy and computational efficiency.

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The method uses the following intermediate slopes:

$$\begin{aligned} k_1 &= h \cdot f(t_n, y_n), \\ k_2 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= h \cdot f(t_n + h, y_n + k_3). \end{aligned}$$

The solution is updated using:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

The Step by Step Method

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

Step 1. We find t_i

$$t_1 = t_0 + h$$

$$t_2 = t_1 + h$$

$$t_3 = t_2 + h$$

$$\vdots$$

$$t_{n+1} = t_n + h$$

Step 2. We find k_i

$$\begin{aligned} k_1 &= h \cdot f(t_n, y_n), \\ k_2 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= h \cdot f(t_n + h, y_n + k_3). \end{aligned}$$

Step 3. We find y_i

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

Example 1.1. Use Rung Kutta method to solve the $D.E.$ and find y_1

$$\frac{dy}{dt} = y - t, \quad \text{with } t_0 = 0, y_0 = 2, h = 0.1$$

Sol. $f(t, y) = y - t$

Step 1. We find t_i: $t_0 = 0$

Step 2 We find k_i when $n = 0$

$$
\begin{aligned}
f(t, y) &= y - t \\
k_1 &= h \cdot f(t_0, y_0) = h \cdot f(0, 2) = 0.1(2 - 0) = 0.2 \\
k_2 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right) = 0.1 \cdot f(0.05, 2.1) \\
&= 0.1(2.1 - 0.05) = 0.205 \\
k_3 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 2 + \frac{0.205}{2}\right) = 0.1 \cdot f(0.05, 2.1025) \\
&= 0.1(2.1025 - 0.05) = 0.2053 \\
k_4 &= h \cdot f(t_0 + h, y_0 + k_3) = 0.1 \cdot f(0 + 0.1, 2 + 0.2053) = 0.1 \cdot f(0.1, 2.2053) \\
&= 0.1(2.2053 - 0.1) = 0.2105
\end{aligned}
$$

Step 3. We find y_1

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) = 2 + \frac{1}{6}(0.2 + 2(0.205 + 0.2053) + 0.2105) \\
&= 2.20693
\end{aligned}
$$

□

Example 1.2. Use Rung Kutta method to solve the $D.E.$ and find y_2

$$\frac{dy}{dt} = ty, \quad \text{with } t_0 = 0.1, y_0 = 4, h = 0.9$$

Sol. $f(t, y) = ty$

Step 1. We find t_i

$$t_0 = 0, \quad t_1 = t_0 + h = 0.1 + 0.9 = 1$$

Step 2 We find k_i when $n = 0$

$$f(t, y) = ty$$

$$k_1 = h \cdot f(0.1, 4) = h \cdot f(0.1, 4) = 0.9(0.1 \times 4) = 0.36$$

$$k_2 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.9 \cdot f\left(0.1 + \frac{0.9}{2}, 4 + \frac{0.36}{2}\right)$$

$$= 0.9 \cdot f(0.55, 4.18) = 0.9(0.55 \times 4.18) = 2.0691$$

$$k_3 = h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.9 \cdot f\left(0.1 + \frac{0.9}{2}, 4 + \frac{2.0691}{2}\right)$$

$$= 0.9 \cdot f(0.55, 5.0346) = 0.9(0.55 \times 5.0346) = 2.4921$$

$$k_4 = h \cdot f(t_0 + h, y_0 + k_3) = 0.9 \cdot f(0.1 + 0.9, 4 + 2.4921)$$

$$= 0.9 \cdot f(1, 6.4921) = 0.9(1 \times 6.4921) = 5.8429$$

Step 3. We find y_1

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) = 4 + \frac{1}{6}(0.36 + 2(2.0691 + 2.4921) + 5.8429)$$

$$= 6.5542$$

Again **(Steps 2,3)** taking $t_1 = 1, y_1 = 6.5542$ to find k_i and y_2.

Step 2 We find k_i when $n = 1$

$$
\begin{aligned}
f(t,y) &= ty \\
k_1 &= h \cdot f(t_1, y_1) = h \cdot f(1, 6.5542) = 0.9(1 \times 6.5542) = 5.8988 \\
k_2 &= h \cdot f\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.9 \cdot f\left(1 + \frac{0.9}{2}, 6.5542 + \frac{5.8988}{2}\right) \\
&= 0.9 \cdot f(1.45, 9.5036) = 0.9(1.45 \times 9.5036) = 12.4022 \\
k_3 &= h \cdot f\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.9 \cdot f\left(1 + \frac{0.9}{2}, 6.5542 + \frac{12.4022}{2}\right) \\
&= 0.9 \cdot f(1.45, 12.7553) = 0.9(1.45 \times 12.7553) = 16.6457 \\
k_4 &= h \cdot f(t_1 + h, y_1 + k_3) = 0.9 \cdot f(1 + 0.9, 6.5542 + 16.6457) \\
&= 0.9 \cdot f(1.9, 23.1999) = 0.9(1.9 \times 23.1999) = 39.6719
\end{aligned}
$$

Step 3. We find y_1

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\
y_2 &= y_2 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\
y_2 &= 6.5542 + \frac{1}{6}(5.8988 + 2(12.4022 + 16.6457) + 39.6719) = 23.832
\end{aligned}
$$

□

Homework of Rung Kutta Method

Use Rung Kutta method to solve the $D.E.$ and find y_3

$$\frac{dy}{dt} = t + y, \quad \text{with } t_0 = 0, y_0 = 0, h = 0.2$$

2 Runge-Kutta-Merson Method

The Runge-Kutta-Merson method is a specific type of Runge-Kutta method that is particularly well-suited for solving ordinary differential equations (ODEs) with a high degree of accuracy. It is a fifth-order method, which means that it is accurate to the fifth order of accuracy.

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The method uses the following intermediate slopes:

$$\begin{aligned}
k_1 &= h \cdot f(t_n, y_n), \\
k_2 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right), \\
k_3 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}\right), \\
k_4 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right), \\
k_5 &= h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4)
\end{aligned}$$

The solution is updated using:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_4 + k_5)$$

The error is estimated as:

$$\text{Error} = \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5)$$

The Step by Step Method

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

Step 1. We find t_i

$$t_1 = t_0 + h$$
$$t_2 = t_1 + h$$
$$t_3 = t_2 + h$$
$$\vdots$$
$$t_{n+1} = t_n + h$$

Step 2. We find k_i

$$k_1 = h \cdot f(t_n, y_n),$$
$$k_2 = h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right),$$
$$k_3 = h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}\right),$$
$$k_4 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right),$$
$$k_5 = h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4)$$

Step 3. We find y_i

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_4 + k_5)$$

Step 4. The error is estimated

$$\text{Error} = \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5)$$

Example 2.1. Use Runge-Kutta-Merson method to solve the $D.E.$ and find y_1

$$\frac{dy}{dt} = t + y, \quad \text{with } t_0 = 0, y_0 = 1, h = 0.1$$

Sol. $f(t, y) = t + y$

Step 1. We find t_i

$$t_0 = 0$$

Step 2 We find k_i when $n = 0$

$$\begin{aligned}
&f(t, y) = t + y \\
&k_1 = h \cdot f(t_0, y_0) = h \cdot f(0, 1) = 0.1(0 + 1) = 0.1 \\
&k_2 = h \cdot f\left(t_0 + \frac{h}{3}, y_0 + \frac{k_1}{3}\right) = h \cdot f\left(0 + \frac{0.1}{3}, 1 + \frac{0.1}{3}\right) \\
&\quad = h \cdot f(0.0333, 1.0333) = 0.1(0.0333 + 1.0333) = 0.1067 \\
&k_3 = h \cdot f\left(t_0 + \frac{h}{3}, y_0 + \frac{k_1}{6} + \frac{k_2}{6}\right) = h \cdot f\left(0 + \frac{0.1}{3}, 1 + \frac{0.1}{6} + \frac{0.1067}{6}\right) \\
&\quad = h \cdot f(0.0333, 1.0345) = 0.1(0.0333 + 1.0345) = 0.1068 \\
&k_4 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right) = h \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{8} + \frac{3(0.1068)}{8}\right) \\
&\quad = h \cdot f(0.05, 1.0526) = 0.1(0.05 + 1.0526) = 0.1103 \\
&k_5 = h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4) \\
&\quad = h \cdot f(0 + 0.1, 1 + \frac{0.1}{2} - \frac{3(0.1068)}{2} + 2(0.1103)) = h \cdot f(0.1, 1.1104) \\
&\quad = 0.1 \cdot (0.1 + 1.1104) = 0.1210
\end{aligned}$$

Step 3. We find y_1

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 4k_4 + k_5) \\ y_1 &= y_0 + \frac{1}{6}(k_1 + 4k_4 + k_5) = 1 + \frac{1}{6}(0.1 + 4(0.1103) + 0.1210) = 1.1104 \end{aligned}$$

Step 4. The error is estimated

$$\begin{aligned} \text{Error} &= \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5) \\ \text{Error} &= \frac{1}{30}(2(0.1) - 9(0.1068) + 8(0.1103) - 0.1210) = 6.6667 \times 10^{-6} \end{aligned}$$

□

Homework of Runge-Kutta-Merson Method

Use Runge-Kutta-Merson method to solve the $D.E.$ and find y_2

$$\frac{dy}{dt} = ty, \quad \text{with } t_0 = 0.1, y_0 = 4, h = 0.9$$