Al-Mustaqbal University

Department of Biomedical Engineering

Third Stage / 1st Course

“Transport Phenomena for BME”

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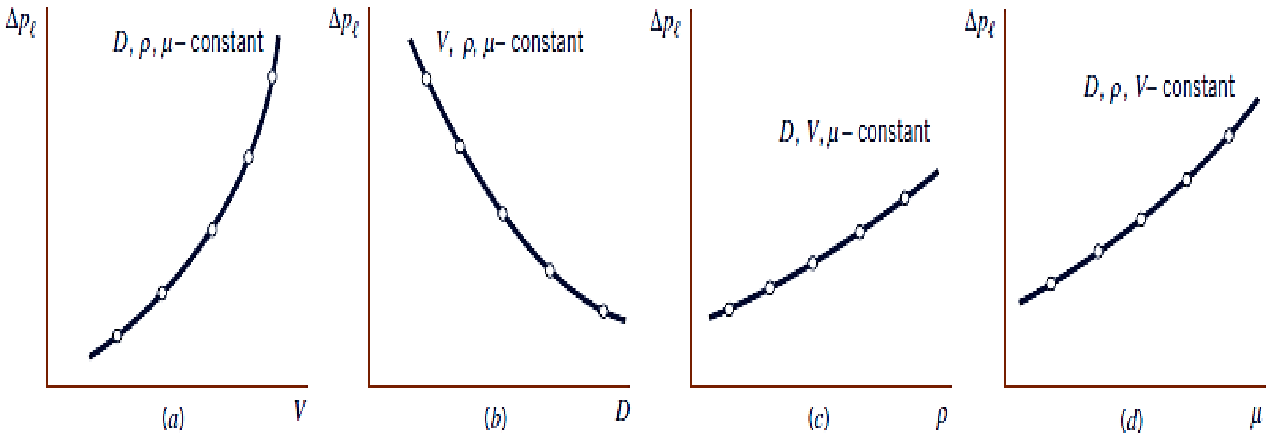
Chapter 4
Dimensional analysis

Dimensional analysis

- To clarify a typical fluid mechanics problem in which experimentation is required, consider the steady flow of an incompressible Newtonian fluid through a **long, smooth walled, horizontal, circular pipe** .
- An important characteristic of this system, which would be of interest to an engineer designing a pipeline, is the **pressure drop per unit length** that develops along the pipe as a result of friction
- The first step in the planning of an experiment to study this problem would be to decide on the factors, or variables, that will have an effect on the pressure drop per unit length, $\Delta p \ell$.
- We expect the list to include the **pipe diameter, *D* the fluid density, *ρ* fluid viscosity, *μ* and the mean velocity, *V*** at which the fluid is flowing through the pipe . Thus, we can express this relationship as
- $\Delta p_\ell = f(D, \rho, \mu, V)$
- At this point, the nature of the function **is unknown**, and the objective of the experiments to be performed is to determine the nature of this function.

Dimensional analysis

- To perform these experiments, it would be necessary **to change one of the variables**, such as the **velocity**, while holding all others constant, and measure the corresponding pressure drop
- This series of tests would yield data that could be represented graphically as is illustrated in Fig a
- It is to be noted that this plot would only be valid for the specific pipe and for the specific fluid used in the tests this **certainly does not give us the general formulation we are looking for**
- We could repeat the process by varying each of the other variables in turn, as is illustrated in Figs b, c, and d



Dimensional analysis

- How would you do this? Finally, once we obtained the various curves shown in Figs a, b, c, and d, how could we combine these data to obtain the desired general functional relationship between $\Delta p_\ell, D, \rho, \mu$ and V which would be valid for any similar pipe system?
- Fortunately, there is a much simpler approach to this problem called **dimensional analysis** that will eliminate such difficulties.
- Dimensional analysis is a mathematical technique used in research work for designing and making model tests
- Dimensional analysis is a method for reducing the number and complexity of experimental variables that affect a given physical phenomena

Dimensional analysis

The three primary purposes of dimensional analysis are

- To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
- To obtain scaling laws so that prototype performance can be predicted from model performance
- To (sometimes) predict trends in the relationship between parameters.
- **Methods of dimensional analysis**
- Rayleigh's method
- Buckingham's Pi-theorem

Scaling Laws are mathematical relationships that describe how certain properties of a system or phenomenon change with alterations in size or scale.

Rayleigh's method (or Power series)

The Rayleigh's method is based on the following steps:

- First of all, write the functional relationship with the given data.
- Write the equation in terms of a constant with exponents i.e. powers a, b, c...
- With the help of the principle of dimensional homogeneity, find out the values of a, b, c by obtaining simultaneous equation and simplify it
- Now substitute the values of these exponents in the main equation, and simplify it.

Example: - Prove that the resistance (F) of a sphere of diameter (D) moving at a constant speed (V) through a fluid density (ρ) and dynamic viscosity (μ) may be expressed as:

- **Solution:**
- Resistance (F), N = [MLT^{-2}]
- Diameter (D), m = [L] ,
- Speed (V), m/s = [LT^{-1}]
- Density (ρ), kg/m^3 = [ML^{-3}]
- Viscosity (μ), kg/m. s = [$ML^{-1}\ T^{-1}$]

$[\text{mass}] = \text{M}$
$[\text{length}] = \text{L}$
$[\text{time}] = \text{T}$
$[\text{area}] = [\text{length} \times \text{length}] = \text{L}^2$
$[\text{volume}] = [\text{area} \times \text{length}] = \text{L}^3$
$[\text{density}] = [\text{mass}/\text{volume}] = \text{M}/\text{L}^3 = \text{ML}^{-3}$
$[\text{velocity}] = [\text{length}/\text{time}] = \text{L}/\text{T} = \text{LT}^{-1}$
$[\text{acceleration}] = [\text{velocity}/\text{time}] = \text{LT}^{-1}/\text{T} = \text{LT}^{-2}$
$[\text{force}] = [\text{mass} \times \text{acceleration}] = \text{MLT}^{-2}$
$[\text{moment of force, torque}] = \text{MLT}^{-2} \times \text{L} = \text{ML}^2\text{T}^{-2}$
$[\text{impulse}] = [\text{force} \times \text{time}] = \text{MLT}^{-2} \times \text{T} = \text{MLT}^{-1}$
$[\text{momentum}] = [\text{mass} \times \text{velocity}] = \text{MLT}^{-1}$
$[\text{work}] = [\text{force} \times \text{distance}] = \text{ML}^2\text{T}^{-2}$
$[\text{kinetic energy}] = [\text{mass} \times (\text{velocity})^2] = \text{ML}^2\text{T}^{-2}$
$[\text{power}] = [\text{work}/\text{time}] = \text{ML}^2\text{T}^{-3}$

Rayleigh's method (or Power series)

$$F = f(D, V, \rho, \mu)$$

$$F = k(D^a, V^b, \rho^c, \mu^d)$$

$$[MLT^{-2}] = [L]^a\,[LT^{-1}]^b\,[ML^{-3}]^c\,[ML^{-1}T^{-1}]^d)$$

For M $\quad 1 = c + d \quad \rightarrow \quad c = 1 - d \quad (1)$

For L $\quad 1 = a + b - 3c - d \quad (2)$

For T $\quad -2 = -b - d \quad \rightarrow \quad b = 2 - d \quad (1)$

By substituting equations (1) and (3) in equation (2) give

$$a = 1 - b + 3c + d$$

$$= 1 - (2 - d) + 3(1 - d) + d$$

$$= 2 - d$$

$$F = k(D^{2-d}, V^{2-d}, \rho^{1-d}, \mu^d)$$

$$F = k(D^2V^2\rho)\left(\frac{\mu}{\rho VD}\right)^d$$

Buckingham's Pi-theorem

- If there are k variables (called dimensional variables or dimensional homogeneous) in a physical phenomenon (in a problem) and if these variables contain r primary or reference dimensions (called fundamental dimensions or fundamental variables, such as M, L, T, then the variables are arranged into (j k r) dimensionless groups (or called dimensionless variables) $\Longrightarrow$ Each dimensionless group is called "pi" (or Π)

- Essentially we assume that for any physically meaningful equation involving k variables, such as

$$u1 = f(\,u2, u3, \ldots\ldots, uk)$$

- The relation among the dimensionless group will be written

  - $\prod 1 = f(\,\prod 2, \prod 3, \ldots\ldots, \prod j{=}k{-}r)$
  - or
  - $\prod 2 = f(\,\prod 1, \prod 3, \ldots\ldots, \prod j{=}k{-}r)$

Several methods can be used to form the dimensionless products, or pi terms, that arise in a dimensional analysis The method we will describe in detail in this section is called the method of repeating variables

Buckingham's Pi-theorem

The Buckingham's Π theorem is based on the following steps:

- Step 1 List all the variables that are involved in the problem and count their total number k
- Step 2 Write down dimensions, express each of the variables in terms of basic dimensions (primary dimensions r
- Step 3 Determine the required number of pi terms (Number of dimensionless groups $\Pi j = k - r$ the expected number of j k r
- Example: -
- Total variable = k = 6
- Fundamental variables = r = 3 (M, L and T)
- Dimensionless variables (j) = $k - r$ = 3 – 6 = 3 No. of dimensionless groups (Π).
- "Step 4. Select a number of repeating variables, these variables will appear in all of our pi group, (usually the number of repeating variables is equal to the
- number of primary dimensions (r)).
- Guidelines for choosing repeating parameters in step 4 of the method of repeating variables
- 1.Never pick variables that are dimensionless.
- 2.The dependent variables should not be selected as repeating variables.
- 3.The chosen repeating must represent all the dimensions in the problem.
- 4.Never pick two parameters with the same dimensions (i.e., variables who both just measure length, like radius of a pipe and height of a pipe).

Buckingham's Pi-theorem

5.When we choose the repeating variables in Pi Buckingham theorem, we can choose based on 3 properties,

- a) Geometry property: (length, width, high and area).
- b) Flow property: (velocity, acceleration, angular velocity, angular acceleration and discharge).
- c) Fluid property: (mass, density, viscosity and surface tension)."

6. General repeater variables (for fluid mechanics)

1. ρ, V, D
2. μ, D, ρ
3. D, Q, ρ

"**Step 5**. Create a pi term (∏ - term) by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent
that will make the combination dimensionless.

Step 6. Repeat Step 5 for each of the remaining nonrepeating variables.

Step 7. Check all the resulting pi terms to make sure they are dimensionless and independent."

1. "$\prod 1 = f(\prod 2, \prod 3, \ , \prod j{=}k{-}r)$"

Determination of Pi Terms

- To illustrate these various steps we will again consider the problem discussed earlier in this lecture which was concerned with the steady flow of an incompressible Newtonian fluid through a long, smooth walled, horizontal circular pipe We are interested in the pressure drop per unit length, along the pipe as illustrated by the figure in the margin

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Step 1 List all the variables that are involved in the problem.

$$\Delta p_\ell = f(D, \rho, \mu, V) \qquad (k = 5)$$

Step 2 Express each of the variables in terms of basic dimensions.

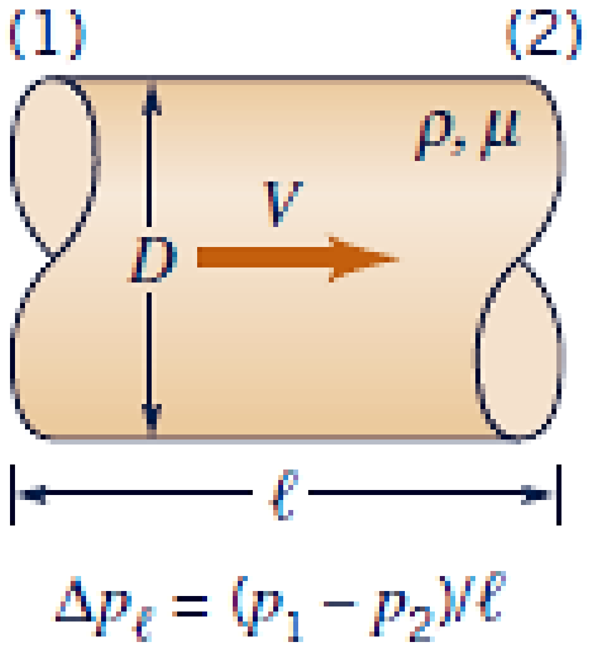
pressure drop per unit length (F), N = $[ML^{-2}T^{-2}]$

Diameter (D), m = [L] ,

Speed (V), m/s = $[LT^{-1}]$

Density (ρ), kg/m^3 = $[ML^{-3}]$

Viscosity (μ), kg/m. s = $[ML^{-1}T^{-1}]$



Step 3 Determine the required number of pi terms.

No.of dimension groups = No.of variables –No.of dimensions

5 variable $\rightarrow$ $\Delta p_\ell, D, \rho, \mu, V$ $(k = 5)$

No.of dimensions = 3 (M L T)

No.of dimension groups = 5–3= **2** ∏ term required

Find the equations for the dimensions

L = D .... (1)

T= L/V .... Put (1) in (2) → T= d/V. .... (3)

M = **ρ** .L^3 .... (4) Put (1) in (4) → M = **ρ** .d^3 .... (5)

$$\prod_1 = \Delta p_\ell * L^2 T^2 / M \text{ ....} (6)$$

Put eq (1) ,(3). And (5) in eq. (6)

$$\prod_1 = \Delta p_\ell * \frac{d^2 . \frac{d^2}{v^2}}{\rho . d^3}$$

$$\prod_1 = \Delta p_\ell * \frac{\frac{d^4}{v^2}}{\rho . d^3}$$

$$\prod_1 = \Delta p_\ell * \frac{d}{\rho . v^2}$$

$$\prod_1 = \frac{M\ L^{-2} T^{-2}\ L}{M\ L^{-3} . L . T^{-1}}$$

$$\prod_1 = M^0 L^0 T^0$$

$$\prod_2 = \mu\ L\, T / M \text{ ....} (7)$$

Put eq (1) ,(3). And (5) in eq. (7)

$$\prod_2 = \mu * \frac{d . \frac{d}{v}}{\rho . d^3}$$

$$\prod_2 = \mu * \frac{\frac{d^2}{v}}{\rho . d^3}$$

$$\prod_2 = \mu * \frac{1}{\rho . v . d}$$

$$\prod_2 = \frac{M\ L^{-1} T^{-1}}{M\ L^{-3} . L . T^{-1} L}$$

$$\prod_2 = M^0 L^0 T^0$$

$$\prod_1 = \prod_2$$

- Step 7 At this point stop and check to make sure the pi terms are actually dimensionless We will check using MLT dimensions Thus,

$$\Pi_1 = M^0L^0T^0$$

$$\Pi_2 = M^0L^0T^0$$

Finally (Step 8), we can express the result of the dimensional analysis as a

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\mu}{DV\rho}\right)$$