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Numerical Analysis

Lecture 13

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1 Fourier series of Even and Odd Function

Fourier series can be used to represent periodic functions as the sum of sines and cosines. The form of the series depends on whether the function is even, odd, or neither.

Definition 1.1 (Even Function). A function $f(x)$ is called an even function if it satisfies the following condition for all x in its domain:

$$f(-x) = f(x).$$

For example, with $f(x) = x^2$:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Thus, $f(x) = x^2$ is an even function.

Definition 1.2 (Odd Function). A function $f(x)$ is called an odd function if it satisfies the following condition for all x in its domain:

$$f(-x) = -f(x).$$

Some common examples of odd functions are:

- $f(x) = x^3$
- $f(x) = \sin(x)$
- $f(x) = x$

For example, with $f(x) = x^3$:

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Thus, $f(x) = x^3$ is an odd function.

1.1 Fourier Series for Even Function

Given a function $f(x)$, the Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos{(nx)} \quad \text{where } b_n = 0$$

Example 1.1. Find Fourier series of the function $f(x) = x^2$, from$x = 0$ to $x = 2\pi$

Sol. To find the Fourier series of $f(x) = x$ defined on the interval $[0, 2\pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Since $f(x) = x^2$ **is even** then $b_n = 0$.

Calculate a_0, a_n

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx = \frac{1}{2\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{(2\pi)^3}{3} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(nx) \, dx$$

Using integration by parts (let $u = x^2$ and $dv = \cos(nx)dx$):

$$u = x^2 \Rightarrow du = 2xdx, \qquad dv = \cos(nx)dx \Rightarrow v = \frac{\sin(nx)}{n}$$

$$\int x^2 \cos(nx)dx = x^2 \cdot \frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n} \cdot 2xdx$$

$$\int x^2 \cos(nx)dx = x^2 \cdot \frac{\sin(nx)}{n} - 2 \int \frac{x \sin(nx)}{n} dx$$

Again using integration by parts on the remaining integral (let $u = x$ and $dv = \sin(nx)dx$):

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = \sin(nx)dx \quad \Rightarrow \quad v = -\frac{\cos(nx)}{n}$$

$$= x^2 \cdot \frac{\sin(nx)}{n} - 2\left(x \cdot -\frac{\cos(nx)}{n^2} - \int -\frac{\cos(nx)}{n^2}dx\right)$$

$$= x^2 \cdot \frac{\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2}{n^2}\int \cos(nx)dx$$

$$= x^2 \cdot \frac{\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2}{n^3}\sin(nx)$$

Evaluating this integral from 0 to 2π:

$$a_n = \frac{1}{\pi}\left[\left(\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right)\Bigg|_0^{2\pi}\right] = -\frac{4}{n^2}$$

The Fourier series of $f(x) = x^2$ is:

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}\cos(nx)$$

□

1.2 Fourier Series for Odd Function

Given a function $f(x)$, the Fourier series is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where } a_0, a_n = 0$$

Example 1.2. Find Fourier series of the function $f(x) = x$, from$x = -\pi$ to $x = \pi$

Sol. To find the Fourier series of $f(x) = x$ defined on the interval $[-\pi, \pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$$

Since $f(x) = x$ **is odd** then $a_0 = a_n = 0$.

Calculate b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx$$

Using integration by parts:

$$\int x \sin(nx)\, dx = -\frac{x \cos(nx)}{n}\Bigg|_{-\pi}^{\pi} + \int \frac{\cos(nx)}{n}\, dx$$

Evaluating this integral:

$$= -\frac{\pi \cos(n\pi)}{n} + \frac{(-\pi) \cos(-n\pi)}{n} + \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= -\frac{2\pi \cos(n\pi)}{n} = -\frac{2\pi(-1)^n}{n}$$

Thus,

$$b_n = \frac{1}{\pi}\left(-\frac{2\pi(-1)^n}{n}\right) = \frac{2(-1)^n}{n}$$

So, the Fourier series for $f(x) = x$ in the interval $[0, 2\pi]$ is:

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin(nx)) \quad \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin(nx)$$

□

2 Half-Range Series

The half-range Fourier series allows us to represent a function using either sines or cosines. This is particularly useful when dealing with functions that are defined only on a finite interval.

2.1 Half-Range Sine Series

For a function $f(x)$, the half-range sine series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (b_n \sin(nx))$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin(nx)\, dx$$

2.2 Half-Range Cosine Series

For a function $f(x)$ defined on $0 \leq x \leq \pi$, the half-range cosine series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where}$$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

Homework 2: Fourier Series

1. Find Fourier series of the function $f(x) = 4x^2$, from$x = 0$ to $x = 2\pi$

2. Find Fourier series of the function

$$f(x) = \begin{cases} 1 \text{ if} & 0 < x < \pi \\ 2 \text{ if} & \pi < x < 2\pi \end{cases}$$

3. Find cosine Half-range series for the function defined as

$$f(x) = x, \quad 0 < x < \pi$$

4. Find sine Half-range series for the function defined as

$$f(x) = x, \quad 0 < x < \pi$$