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Numerical Analysis

Lecture 1

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1 Introduction to Numerical Analysis

Numerical analysis is the study of algorithms used to solve mathematical problems that are difficult or impossible to solve analytically. These algorithms are employed to provide approximate solutions to problems in various fields such as engineering, physics, economics, and computer science.

The primary objectives of numerical analysis include:

- Developing efficient algorithms.
- Estimating the error of approximate solutions.
- Ensuring the stability of computations.
- Solving equations, both linear and non-linear.
- Differentiation and integration of functions.
- Solving systems of equations.
- Solving differential equations.

1.1 Computers and Numerical Analysis

$$Numerical\ Methods + Program\ Computers = Numerical\ Analysis$$

- As you will learn enough about many numerical methods, you will be able to write programs to implement them.
- Programs can be written in any computer language. In this course all programs will be written in Matlab environment.

1.2 Types of Errors in Numerical Analysis

Numerical methods are approximate by nature, and errors are introduced at various stages. The most common types of errors are:

- **Round-off Error**: Caused by the limited precision with which computers store real numbers.
- **Truncation Error**: Arises when an infinite process (such as a series expansion) is approximated by a finite number of terms.
- **Absolute Error**: The difference between the exact value and the approximate value.

$$\text{Absolute Error} = |x_{\text{exact}} - x_{\text{approx}}|$$

- **Relative Error**: The absolute error divided by the magnitude of the exact value, giving a measure of the error relative to the exact quantity.

$$\text{Relative Error} = \frac{|x_{\text{exact}} - x_{\text{approx}}|}{|x_{\text{exact}}|}$$

Illustrative Example: Round-off Error

Consider the subtraction of two nearly equal numbers using limited precision. Let:

$$x = 1.234567 \quad \text{and} \quad y = 1.234564$$

Their exact difference is:

$$x - y = 0.000003$$

However, if both x and y are stored using only four decimal places, we have:

$$x = 1.2346 \quad \text{and} \quad y = 1.2346$$

Thus, the difference becomes:

$$x - y = 0$$

This demonstrates a loss of precision due to round-off error.

Illustrative Example: Truncation Error

Consider the Taylor series expansion of e^x around $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If we approximate e^x by truncating the series after the first two terms, the approximation becomes:

$$e^x \approx 1 + x$$

For small x, this approximation may be reasonable, but for larger x, the truncation error increases.

For example, when $x = 1$:

$$e^1 = 2.71828 \quad \text{and} \quad 1 + 1 = 2$$

The truncation error is:

$$\text{Truncation Error} = |2.71828 - 2| = 0.71828d$$

2 Solution of Non-linear Equations

Non-linear equations are equations of the form:

$$f(x) = 0$$

where $f(x)$ is a non-linear function. In many cases, these equations cannot be solved analytically, and numerical methods are required to approximate the solution. There are various techniques used to solve non-linear equations, such as the Newton-Raphson method, Bisection method, and Secant method.

2.1 Newton-Raphson Method

The Newton-Raphson method is an iterative technique for finding approximate solutions to real-valued functions. It's particularly useful for finding roots of equations.

Method Description

Let $f, f' \in [a, b]$ are continuous function on $[a, b]$. The formula for the Newton-Raphson iteration is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

where:

- x_n is the current approximation,
- $f(x_n)$ is the function evaluated at x_n,
- $f'(x_n)$ is the derivative of the function evaluated at x_n.

Remark 2.1. $\{x_n\}$ be a sequence of approximate roots for f, such that:

$$f'(x) \neq 0, \quad |x_n - \alpha| < \epsilon$$

Where, α is the exact root for $f(x) = 0$.

Remark 2.2. In order to guarantee that, the iterative process is convergent, the initial root, x_0, should be chosen close to the exact root α, which means:

$$|x_0 - \alpha| < \epsilon$$

Steps of Newton-Raphson Algorithm:

Step 1 Choose appropriate initial root $x_0 \in [a, b]$.

Step 2 Let $f(x)$ be the function for which you want to find the root. Compute the derivative $f'(x)$.

Step 3 Calculate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Step 4 Set $n = n + 1$, and continue in the iterative processes,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until we get the stop condition is satisfied: $E_n = |x_{n+1} - x_n| < \epsilon$

Examples of Newton-Raphson Algorithm

Example 2.1. Use Newton-Raphson algorithm to find the approximate root of the following equation:

$$f(x) = x^2 - 2 = 0$$

and stop when $E_n = |x_{n+1} - x_n| = 0$, where $x_0 = 1.5$. Also, find the iterative error at each step.

Solution.

$$f(x) = x^2 - 2, \quad f'(x) = 2x, \quad x_0 = 1.5$$

$$f(x_0) = f(1.5) = (1,5)^2 - 2 = 0.25$$

$$f'(x_0) = f'(1.5) = 2(1.5) = 3$$

1st iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$x_1 = 1.5 - \frac{0.25}{3}$$
$$x_1 = 1.4167$$

2nd iteration

$$f(x_1) = f(1.4167) = (1.4167)^2 - 2 = 0.007$$
$$f'(x_1) = f'(1.4167) = 2(1.4167) = 2.8334$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_2 = 1.4167 - \frac{0.007}{2.8334}$$
$$x_2 = 1.4142$$

The iterative error $E_1 = |x_2 - x_1| = |1.4142 - 1.4167| = 0.0025$

3rd iteration

$$f(x_2) = f(1.4142) = (1.4142)^2 - 2 = 0$$
$$f'(x_2) = f'(1.4142) = 2(1.4142) = 2.8284$$
$$x_3 = 1.4142 - \frac{f(x_2)}{f'(x_2)}$$
$$x_2 = 1.4142 - \frac{0}{2.8284}$$
$$x_2 = 1.4142$$
$$E_2 = |x_3 - x_2| = |1.4142 - 1.4142| = 0$$

Approximate root of the equation $x^2 - 2 = 0$ using Newton Raphson method is 1.4142 (After 3 iterations) □

Example 2.2. Use Newton-Raphson algorithm to find the approximate root of the following equation:

$$f(x) = \frac{1}{x} + 1 = 0$$

with $x_0 = -0.5$ for three iterative steps (only find x_1, x_2, x_3). Also, find the Relative percent error at each step, where $e_n = |\frac{x_{n+1}-x_n}{x_{n+1}}| \times 100\%$

Solution.

$$f(x) = \frac{1}{x} + 1, \quad f'(x) = -\frac{1}{x^2}, \quad x_0 = -0.5$$
$$f(x_0) = f(-0.5) = \frac{1}{-0.5} + 1 = -1$$
$$f'(x_0) = f'(-0.5) = -\frac{1}{(-0.5)^2} = -4$$

1st iteration

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.5 - \frac{-1}{-4} = -0.75$$

2nd iteration

$$f(x_1) = f(-0.75) = \frac{1}{-0.75} + 1 = -0.3333$$
$$f'(x_1) = f'(-0.75) = -\frac{1}{(-0.75)^2} = -1.7778$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.75 - \frac{-0.3333}{-1.7778} = -0.9375$$

Relative percent error $e_1 = |\frac{x_2 - x_1}{x_2}| \times 100\% = |\frac{-0.9375 - (-0.75)}{-0.9375}| \times 100\% = 20\%$

3rd iteration

$$
\begin{aligned}
f(x_2) &= f(-0.9375) = \frac{1}{-0.9375} + 1 = -0.0667 \\
f'(x_2) &= f'(-0.9375) = -\frac{1}{(-0.9375)^2} = -1.7778 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = -0.9375 - \frac{-0.06670}{-1.7778} = -0.9750 \\
e_2 &= |\frac{x_3 - x_2}{x_3}| \times 100\% = |\frac{-0.9750 - (-0.9375)}{-0.9750}| \times 100\% = 3.8462\%
\end{aligned}
$$

□

Homework of Newton-Raphson Algorithm

1. Use Newton-Raphson algorithm to find the approximate root of the following equation: $f(x) = x \sin x - 1 = 0$, with $x_0 = 1$ for three iterative steps (only find x_1, x_2, x_3). Also, find the iterative error.

2. Use Newton-Raphson algorithm to find the approximate root of the following equation: $f(x) = x^2 - 6 = 0$, with $x_0 = 1$ for two iterative steps (only find x_1, x_2). Also, find the Relative percent error at each step.

3. Find a root of the equation $x \sin x + \cos x$, for three iterative steps (only find x_1, x_2, x_3), where $x_0 = -1$.
   Hint: $f(x) = x \sin x + \cos x,\ f'(x) = x \cos x$

3 Interpolation and Approximation

3.1 Lagrange Approximation

Lagrange Interpolation is a way of finding the value of any function at any given point when the function is not given. We use other points on the function to get the value of the function at any required point.

If we are given $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, the Lagrange polynomial $P(x)$ is defined as:

$$P(x) = \sum_{i=0}^{n} y_i \ell_i(x) \quad \text{where } \ell_i(x) \text{ are the Lagrange basis polynomials defined by:}$$

$$\ell_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}$$

$$\ell_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_j)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_j)}$$

For example: For 3 data points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, the Lagrange polynomial is:

$$P(x) = y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x)$$

$$P(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Examples of the Lagrange Approximation

Example 3.1. Find $f(2.7)$ using Lagrange's Interpolation formula

x_i	2	2.5	3
y_i	0.69315	0.91629	1.09861

Solution.

$$\ell_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2.5)(x - 3)}{(2 - 2.5)(2 - 3)} = \frac{x^2 - 5.5x + 7.5}{0.5}$$

$$\ell_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 2)(x - 3)}{(2.5 - 2)(2.5 - 3)} = \frac{x^2 - 5x + 6}{-0.25}$$

$$\ell_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 2)(x - 2.5)}{(3 - 2)(3 - 2.5)} = \frac{x^2 - 4.5x + 5}{0.5}$$

$P(x) = y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x)$

$P(x) = (0.69315)(\frac{x^2 - 5.5x + 7.5}{0.5}) + (0.91629)(\frac{x^2 - 5x + 6}{-0.25}) + (1.09861)(\frac{x^2 - 4.5x + 5}{0.5})$

$P(x) = (1.3863)(x^2 - 5.5x + 7.5) + (-3.66516)(x^2 - 5x + 6) + (2.1972)(x^2 - 4.5x + 5)$

$P(x) = (1.3863x^2 - 7.6247x + 10.3973) + (-3.66516x^2 + 18.3258x - 21.99096)$

$+ (2.1972x^2 - 9.8874x + 10.9861)$

$P(x) = -0.08166x^2 + 0.8137x - 0.60756$

$x = 2.7 \Rightarrow f(2.7) = P(2.7) = -0.08166x(2.7)^2 + 0.8137(2.7) - 0.60756 = 0.9941$ □

Example 3.2. Find the Lagrange interpolation polynomial that takes the values $f(0) = 1, f(1) = 1, f(2) = 2, f(4) = 5.$

Solution. Given the points: $f(0) = 1, \quad f(1) = 1, \quad f(2) = 2, \quad f(4) = 5$

The Lagrange basis polynomials $\ell_i(x)$ are:

$$\begin{aligned}
\ell_0(x) &= \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)} = \frac{x^3 - 7x^2 + 14x - 8}{-8} \\
\ell_1(x) &= \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} = \frac{(x)(x-2)(x-4)}{(1)(-1)(-3)} = \frac{x^3 - 6x^2 + 8x}{3} \\
\ell_2(x) &= \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} = \frac{(x)(x-1)(x-4)}{(2)(1)(-2)} = \frac{x^3 - 6x^2 + 4x}{-4} \\
\ell_3(x) &= \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} = \frac{(x)(x-1)(x-2)}{(4)(3)(2)} = \frac{x^3 - 3x^2 + 2x}{24}
\end{aligned}$$

The Lagrange interpolation polynomial is:

$$
\begin{aligned}
P(x) &= f_0\ell_0(x) + f_1\ell_1(x) + f_2\ell_2(x) + f_3\ell_3(x)\\
P(x) &= (1)(\frac{x^3-7x^2+14x-8}{-8}) + (1)(\frac{x^3-6x^2+8x}{3}) + (2)(\frac{x^3-6x^2+4x}{-4})\\
&\quad + (5)(\frac{x^3-3x^2+2x}{24})\\
P(x) &= (-0.125)(x^3-7x^2+14x-8) + (0.3333)(x^3-6x^2+8x)\\
&\quad + (-0.5)(x^3-6x^2+4x) + (0.2083)(x^3-3x^2+2x)\\
P(x) &= (-0.125x^3+0.875x^2-1.75x+1) + (0.3333x^3-2x^2+2.6667x)\\
&\quad + (-0.5x^3+3x^2-2x) + (0.2083x^3-0.625x^2+0.4167x)
\end{aligned}
$$

$\Rightarrow P(x) = -0.0833x^3 + 1.25x^2 - 0.6667x + 1$

H.W: Find $P(3)$ □

Homework of Lagrange Approximation

1. Find $f(150)$ using Lagrange's Interpolation formula

x_i	300	304
y_i	2.4829	2.4771

2. If $y(1) = 12$, $y(2) = 15$, $y(5) = 25$, and $y(6) = 30$. Find the four points Lagrange interpolation polynomial that takes some value of function (y) at the given points and estimate the value of $y(4)$ at given points.

3. Fit a cubic through the first four points $y(3.2) = 22.0$, $y(2.7) = 17.8$, $y(1.0) = 14.2$,and $y(5.6) = 51.7$, to find the interpolated value for $x = 3.0$ function (y) at the given points and estimate the value of $y(4)$ at given points.

4. If $f(1.0) = 0.7651977$, $f(1.3) = 0.6200860$, $f(1.6) = 0.4554022$, $f(1.9) = 0.2818186$ and $f(2.2) = 0.1103623$. Use Lagrange polynomial to approximation to $f(1.5)$.