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Numerical Analysis

Lecture 2

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الرياضيات المتقدمة: المرحلة الثانية

مادة التحليل العددي

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Contents

1 Numerical Integration

Integration is the process of measuring the area under a function plotted on a graph. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods have been developed to find the integral.

Numerical integration refers to techniques used to approximate the value of definite integrals. Instead of solving an integral analytically, these methods estimate the integral using a finite sum of values of the function at specific points. There some types of integration which are Trapezoidal rule, Simpsons 1/3 rule, and Simpsons 3/8 rule.

1.1 Trapezoidal Method

The Trapezoidal Method is a numerical integration technique used to approximate the value of a definite integral. It works by dividing the area under the curve into a series of trapezoids rather than rectangles, which often results in a better approximation than a simple Riemann sum.

Trapezoidal Rule

If you want to approximate the integral of a function $f(x)$ over the interval $[a, b]$, the Trapezoidal Rule can be expressed as:

$$\begin{aligned} I = \int_a^b f(x)\,dx &\approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] \\ &\approx \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)\right] \end{aligned}$$

such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$h = \frac{b-a}{n}$, and $x_i = x_{i-1} + h$ are the points at which the function is evaluated.

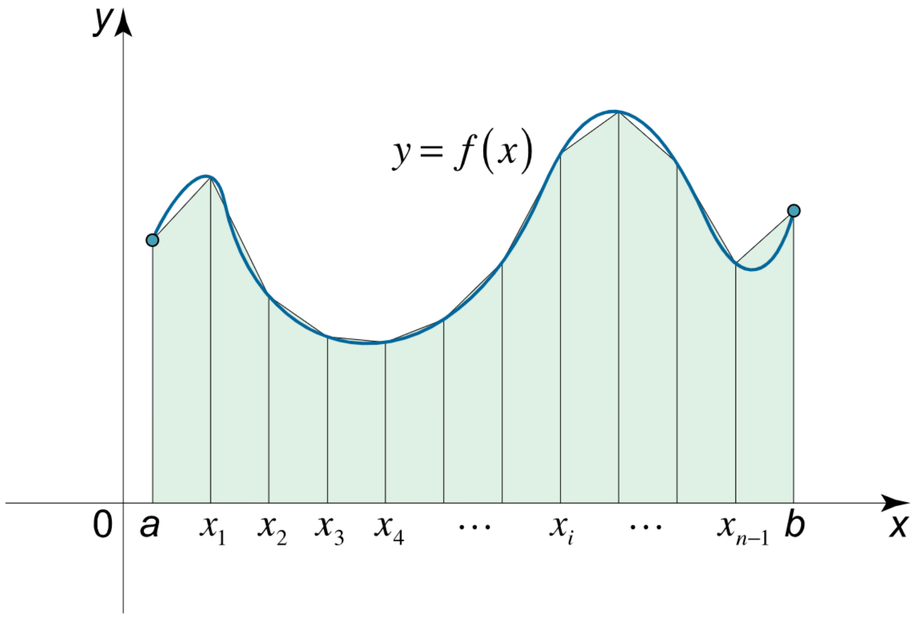

Figure 1: Trapezoidal Method

Examples of Trapezoidal Rule

Example 1.1. Let us approximate the integral $\int_0^2 e^{x^2}\,dx$ using the Trapezoidal Rule with $n = 5$ intervals.

Solution. The interval is $[0, 2]$, and we are dividing it into $n = 5$ subintervals.

Step 1: Calculate h

$$h = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$$

Step 2: Determine the evaluation points x_i, since $n = 5 \Rightarrow x_0, x_1.x_2, x_3, x_4, x_5$.

$$x_0 = a = 0$$

$$x_1 = x_0 + h = 0 + 0.4 = 0.4$$

$$x_2 = x_1 + h = 0.4 + 0.4 = 0.8$$

$$x_3 = x_2 + h = 0.8 + 0.4 = 1.2$$

$$x_4 = x_3 + h = 1.2 + 0.4 = 1.6$$

$$x_5 = x_4 + h = 1.6 + 0.4 = 2 = b$$

Step 3: Evaluate the function at these points

$$f(x_0) = e^{x_0^2} = f(0) = e^{0^2} = 1$$

$$f(x_1) = e^{x_1^2} = f(0.4) = e^{(0.4)^2} = e^{0.16} \approx 1.1735$$

$$f(x_2) = e^{x_2^2} = f(0.8) = e^{(0.8)^2} = e^{0.64} \approx 1.8965$$

$$f(x_3) = e^{x_3^2} = f(1.2) = e^{(1.2)^2} = e^{1.44} \approx 4.2207$$

$$f(x_4) = e^{x_4^2} = f(1.6) = e^{(1.6)^2} = e^{2.56} \approx 12.9358$$

$$f(x_5) = e^{x_5^2} = f(2) = e^{(2)^2} = e^{4} \approx 54.5982$$

Step 4: Apply the Trapezoidal Rule

$$\begin{aligned} I = \int_a^b f(x)\,dx = \int_0^2 e^{x^2}\,dx &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right] \\ &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + f_4) + f_5\right] \\ &\approx \frac{0.4}{2}\left[1 + 2(1.1735 + 1.8965 + 4.2207 + 12.9358) + 54.5982\right] \\ &\approx 0.2\left[1 + 2(20.2265) + 54.5982\right] = 0.2\left[96.0512\right] = 19.2102 \end{aligned}$$

$$\therefore I \approx 19.2102$$

$\square$

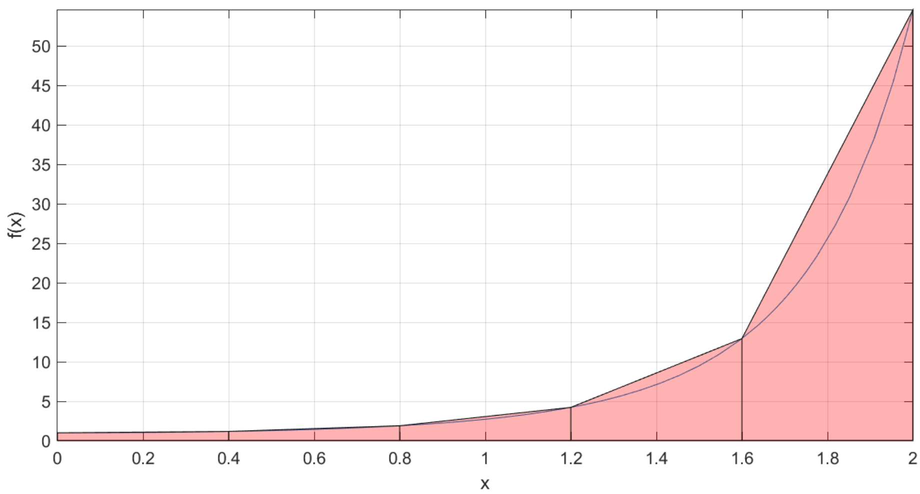

Figure 2: Trapezoidal Method Approximation for $f(x) = e^{x^2}$

Example 1.2. Evaluate the integral $\ln(x)$ by trapezoidal rule dividing the interval $[1, 2]$ into four equal parts.

Solution. The interval is $[1, 2]$, and we are dividing it into $n = 4$ subintervals.

Step 1: Calculate h

$$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

Step 2: Determine the evaluation points x_i, since $n = 4 \Rightarrow x_0, x_1.x_2, x_3, x_4$.

$$x_0 = a = 1, \quad x_1 = x_0 + h = 1 + 0.25 = 1.25$$

$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$

$$x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$

$$x_4 = x_3 + h = 1.75 + 0.25 = 2 = b$$

Step 3: Evaluate the function at these points

$$f_0 = \ln(x_0) = f(1) = \ln(1) = 0, \quad f_1 = \ln(x_1) = f(1.25) = \ln(1.25) \approx 0.2231$$

$$f_2 = \ln(x_2) = f(1.5) = \ln(1.5) \approx 0.4055$$

$$f_3 = \ln(x_3) = f(1.75) = \ln(1.75) \approx 0.5596, f_4 = \ln(x_4) = f(2) = \ln(2) \approx 0.6931$$

Step 4: Apply the Trapezoidal Rule

$$\begin{aligned} I = \int_a^b f(x)\,dx &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right] \\ &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right] \\ &\approx \frac{0.25}{2}\left[0 + 2(0.2231 + 0.4055 + 0.5596) + 0.6931\right] \\ &\approx 0.125\left[0 + 2(1.1882) + 0.6931\right] = 0.125\left[3.0695\right] = 0.3837 \end{aligned}$$

$$\therefore I \approx 0.3837$$

□

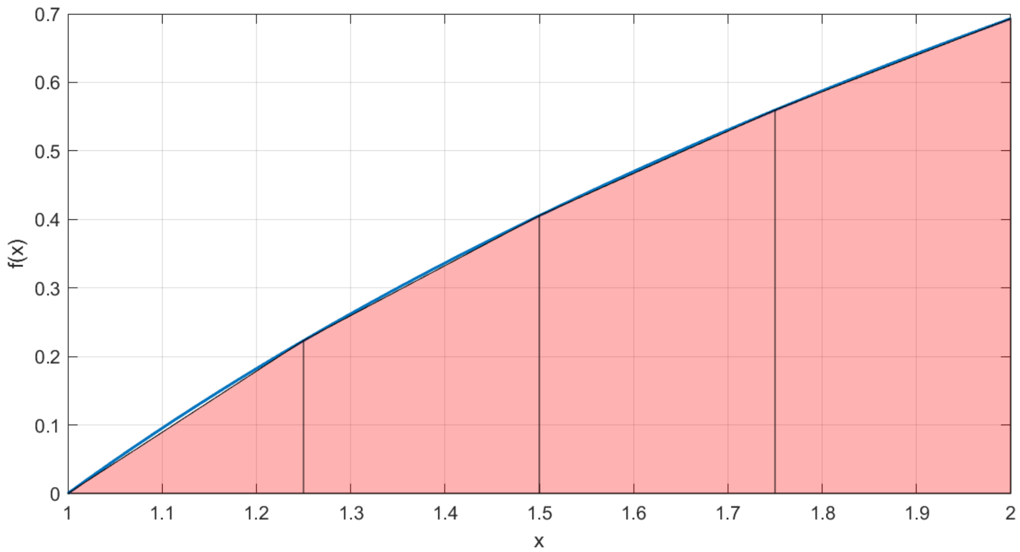

Figure 3: Trapezoidal Method Approximation for $f(x) = \ln(x)$

Homework of Trapezoidal Rule

1. Approximate $\int_0^1 \sin(x^2)dx$, by the trapezoidal rule, with $n = 6$.
2. Evaluate the integral $\sqrt{x^3+1}$ by trapezoidal rule dividing the interval $[0, 3]$ into five equal parts.

1.2 Simpsons Method

The Simpsons method is another numerical for approximating the definite integral of a function. It generally provides a more accurate approximation than the Trapezoidal Rule for the same number of intervals.

Simpsons 1/3 Rule

If you want to approximate the integral of a function $f(x)$ over the interval $[a, b]$, the Simpsons 1/3 Rule can be expressed as:

$$I = \int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right]$$

such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$h = \frac{b-a}{n}$, and $x_i = x_{i-1} + h$ are the points at which the function is evaluated.

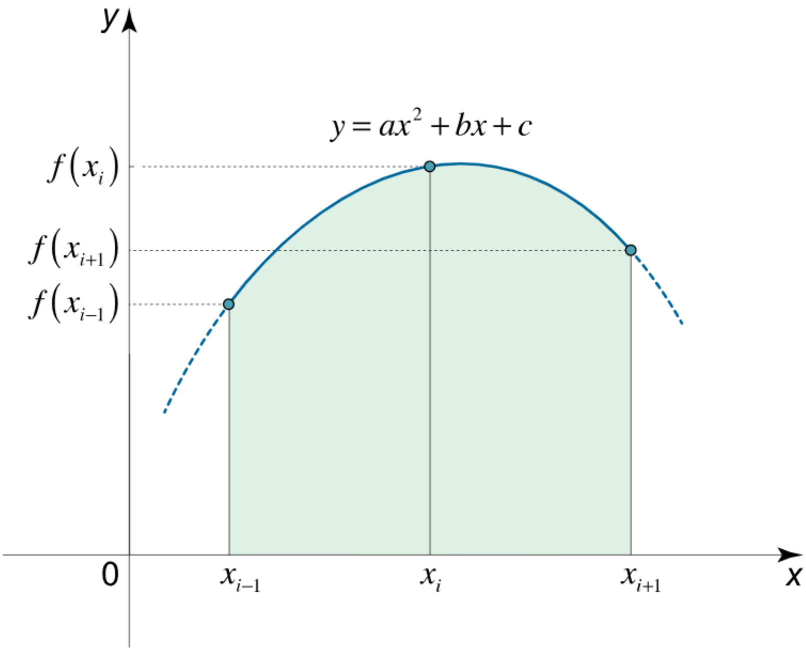

Figure 4: Simpsons 1/3 Method

Examples of Simpsons 1/3 Rule

Example 1.3. Let us approximate the integral $\int_0^2 e^{x^2}\,dx$ using the Simpsons 1/3 Rule with $n = 5$ intervals.

Solution. The interval is $[0, 2]$, and we are dividing it into $n = 5$ subintervals.

Step 1: Calculate h

$$h = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$$

Step 2: Determine the evaluation points x_i, since $n = 5 \Rightarrow x_0, x_1.x_2, x_3, x_4, x_5$.

$$\begin{aligned}
x_0 &= a = 0 \\
x_1 &= x_0 + h = 0 + 0.4 = 0.4 \\
x_2 &= x_1 + h = 0.4 + 0.4 = 0.8 \\
x_3 &= x_2 + h = 0.8 + 0.4 = 1.2 \\
x_4 &= x_3 + h = 1.2 + 0.4 = 1.6 \\
x_5 &= x_4 + h = 1.6 + 0.4 = 2 = b
\end{aligned}$$

Step 3: Evaluate the function at these points

$$\begin{aligned}
f(x_0) &= e^{x_0^2} = f(0) = e^{0^2} = 1 \\
f(x_1) &= e^{x_1^2} = f(0.4) = e^{(0.4)^2} = e^{0.16} \approx 1.1735 \\
f(x_2) &= e^{x_2^2} = f(0.8) = e^{(0.8)^2} = e^{0.64} \approx 1.8965 \\
f(x_3) &= e^{x_3^2} = f(1.2) = e^{(1.2)^2} = e^{1.44} \approx 4.2207 \\
f(x_4) &= e^{x_4^2} = f(1.6) = e^{(1.6)^2} = e^{2.56} \approx 12.9358 \\
f(x_5) &= e^{x_5^2} = f(2) = e^{(2)^2} = e^{4} \approx 54.5982
\end{aligned}$$

Step 4: Apply the Simpsons 1/3 Rule

$$\begin{aligned}
I = \int_a^b f(x)\,dx &\approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right] \\
&\approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3) + 2(f_2 + f_4) + f_5\right] \\
&\approx \frac{0.4}{3}\left[1 + 4(1.1735 + 4.2207) + 2(1.8965 + 12.9358) + 54.5982\right] \\
&\approx \frac{0.4}{3}\left[1 + 21.5768 + 29.6646 + 54.5982\right] = \frac{0.4}{3}\left[106.8396\right]
\end{aligned}$$

$$\therefore I \approx 14.2453$$

□

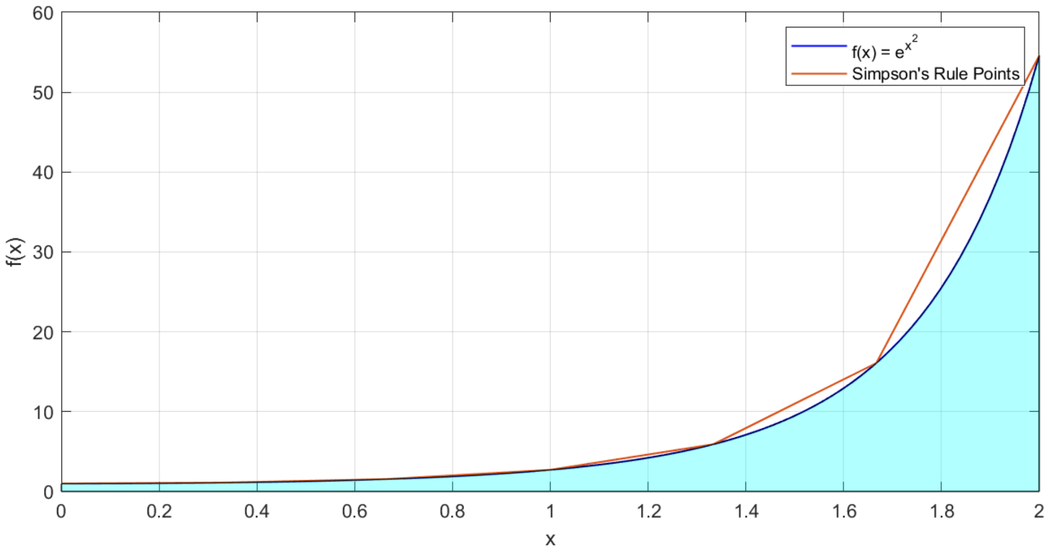

Figure 5: Simpsons 1/3 Method Approximation for $f(x) = e^{x^2}$

Example 1.4. Evaluate the integral $\ln(x)$ by Simpsons 1/3 rule dividing the interval $[1, 2]$ into four equal parts.

Solution. The interval is $[1, 2]$, and we are dividing it into $n = 4$ subintervals.

Step 1: Calculate h

$$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

Step 2: Determine the evaluation points x_i, since $n = 4 \Rightarrow x_0, x_1.x_2, x_3, x_4$.

$$x_0 = a = 1, \quad x_1 = x_0 + h = 1 + 0.25 = 1.25$$

$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$

$$x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$

$$x_4 = x_3 + h = 1.75 + 0.25 = 2 = b$$

Step 3: Evaluate the function at these points

$$f_0 = \ln(x_0) = f(1) = \ln(1) = 0, \quad f_1 = \ln(x_1) = f(1.25) = \ln(1.25) \approx 0.2231$$

$$f_2 = \ln(x_2) = f(1.5) = \ln(1.5) \approx 0.4055$$

$$f_3 = \ln(x_3) = f(1.75) = \ln(1.75) \approx 0.5596, f_4 = \ln(x_4) = f(2) = \ln(2) \approx 0.6931$$

Step 4: Apply the Simpsons 1/3 Rule

$$\begin{aligned} I = \int_a^b f(x)\,dx &\approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right] \\ &\approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3) + 2(f_2) + f_4\right] \\ &\approx \frac{0.25}{3}\left[0 + 4(0.2231 + 0.5596) + 2(0.4055) + 0.6931\right] \\ &\approx \frac{0.25}{3}\left[0 + 3.1308 + 0.811 + 0.6931\right] = \frac{0.25}{3}\left[4.6349\right] = 0.3862 \end{aligned}$$

□

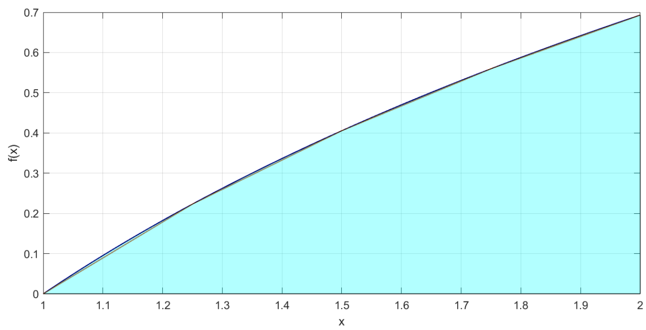

Figure 6: Simpsons 1/3 Method Approximation for $f(x) = \ln(x)$

Homework of Simpsons 1/3 Rule

1. Approximate $\int_0^1 \sin(x^2)dx$, by the Simpsons 1/3 rule, with $n = 6$.
2. Evaluate the integral $\sqrt{x^3+1}$ by Simpsons 1/3 rule dividing the interval $[0, 3]$ into five equal parts.