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# Numerical Analysis

## Lecture 8

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# 1 Heat Equation

The heat equation is a fundamental partial differential equation that describes how heat (or temperature) evolves over time in a given region. It is widely used in physics, engineering, and mathematics to model heat conduction.

**The One-Dimensional Heat Equation**

In its simplest form, the one-dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ is the temperature distribution function, representing the temperature at position $x$ and time $t$.
- $k$ is the thermal diffusivity constant, which depends on the material properties.
- $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with respect to time.
- $\frac{\partial^2 u}{\partial x^2}$ represents the spatial second derivative of temperature, indicating how temperature varies along the spatial axis.

**Solution of the Heat Equation**

The method of separation of variables is commonly used to solve the heat equation, leading to solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substituting $u(x,t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

**Example 1.1** (Heat Equation)**.** Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0,t) = 0, u(10,t) = 0 \tag{2}$$

*Sol.* To solve Eq. (1) suppose $u(x,t) = XT$.

Then

$$\frac{\partial u}{\partial t} = XT' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Since $u(0,t) = 0, u(10,t) = 0 \Rightarrow X(0) = 0, X(10) = 0$

Put in Eq. (1)

$$XT' = 2X''T$$

$$\frac{T'}{2T} = \frac{X''}{X}$$

Now let

$$\frac{T'}{2T} = \frac{X''}{X} = b, \quad b \text{ constant}$$

$$\frac{T'}{2T} = b \Rightarrow T' = 2bT$$

$$\frac{X''}{X} = b \Rightarrow X'' = 2bX \Rightarrow X'' - 2bX = 0$$

Now let $b = \lambda^2$

when $\lambda^2 \geq 0$ trivial solution. Then $b = -\lambda^2 < 0$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Since $X(0) = 0$,

$$\Rightarrow X(0) = A\cos(\lambda(0)) + B\sin(\lambda(0)) = A \Rightarrow A = 0,$$

$$X(10) = A\cos(\lambda(10)) + B\sin(\lambda(10)) = 0 \Rightarrow X(10) = B\sin(\lambda(10))$$

$$\therefore \sin(\lambda(10)) = 0$$

Since $\sin(\lambda(10)) = 0 \Rightarrow 10\lambda = n\pi, n = 0, 1, 2, \ldots$

$$\Rightarrow \lambda = \frac{n\pi}{10}$$

$$\therefore x(x) = B_n \sin(\frac{n\pi}{10}x)$$

Since $T' = 2bT \Rightarrow T' = e^{2bt} \Rightarrow T' = e^{-2\lambda^2 t} \Rightarrow T' = e^{-2(\frac{n\pi}{10})^2 t}$

$$\Rightarrow T = C_n e^{-2(\frac{n\pi}{10})^2 t}$$

$$\therefore u(x,t) = XT = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{10}x) e^{-2(\frac{n\pi}{10})^2 t}, A_n = B_n C_n$$

□

**Homework of Heat Equation**

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = 0, \quad k \text{ constant}$$

With boundary condition

$$u(0,t) = 0, u(L,t) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 6\frac{\partial^2 u}{\partial x^2} = 0$$

With boundary condition

$$u(0,t) = 0, u(30,t) = 0$$

# 2 Wave Equation

The wave equation is a second-order partial differential equation that describes the propagation of waves, such as sound waves, light waves, or water waves, in a given medium. It is a fundamental equation in physics and engineering, with applications in fields such as acoustics, electromagnetism, and fluid dynamics.

**The Wave Equation in One Dimension**

In one-dimensional space, the wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ represents the displacement of the wave at position $x$ and time $t$,
- $c$ is the speed of wave propagation in the medium.
- Second Partial Derivative with Respect to Time ($\frac{\partial^2 u}{\partial t^2}$):
  - This term represents the **acceleration** of the wave function $u(x,t)$ with respect to time at any given position $x$.

  - It indicates how the wave's displacement changes over time, capturing the oscillatory nature of waves.

- Second Partial Derivative with Respect to Space ($\frac{\partial^2 u}{\partial x^2}$):

  - This term measures the **curvature** of the wave function $u(x, t)$ with respect to spatial dimensions.
  - Physically, it represents how the wave's displacement changes along the spatial dimension(s), indicating how the wave bends or curves at any given point.

**Solution of the Wave Equation**

The method of separation of variables is commonly used to solve the wave equation, leading to solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substituting $u(x,t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

**Example 2.1.** Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0,t) = 0, u(4,t) = 0, u(x,0) = 5\sin(\frac{\pi}{4}x), u_t(x,0) = 0 \tag{2}$$

*Sol.* To solve Eq. (1) suppose $u(x,t) = XT$.

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Put in Eq. (1)

$$XT'' = 4X''T \Rightarrow \frac{T''}{4T} = \frac{X''}{X}$$
$$\frac{T''}{4T} = \frac{X''}{X} = -p^2, \quad p^2 \text{ constant}$$

Now let

$$\frac{T''}{4T} = -p^2 \Rightarrow T'' + 4p^2T = 0$$
$$\frac{X''}{X} = -p^2 \Rightarrow X'' + 4p^2X = 0$$

$$\Rightarrow X(x) = C_1 \cos(px) + C_2 \sin(px)$$

$$T(t) = C_3 \cos(pt) + C_4 \sin(pt)$$

$$\Rightarrow u(x,t) = (C_1 \cos(px) + C_2 \sin(px))(C_3 \cos(pt) + C_4 \sin(pt))$$

Since $u(0,t)$,

$$\Rightarrow u(0,t) = C_1(C_3 \cos(pt) + C_4 \sin(pt)) = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow u(x,t) = C_2 \sin(px)(C_3 \cos(pt) + C_4 \sin(pt))$$

Since $u(0,4)$,

$$\Rightarrow u(4,t) = C_2 \sin(4p)(C_3 \cos(pt) + C_4 \sin(pt)) = 0$$

Since $C_2 \neq 0 \Rightarrow \sin(4p) = 0 \Rightarrow 4p = n\pi \Rightarrow p = \frac{n\pi}{4}, n = 0, 1, \cdots$

$$\Rightarrow u(x,t) = C_2 \sin(\frac{n\pi}{4}x)(C_3 \cos(\frac{n\pi}{4}t) + C_4 \sin(\frac{n\pi}{4}t))$$

$$u_t(x,t) = C_2 \sin(\frac{n\pi}{4}x)(-\frac{n\pi}{4}C_3 \sin(\frac{n\pi}{4}t) + \frac{n\pi}{4}C_4 \cos(\frac{n\pi}{4}t))$$

Since $u_t(x,0) = 0$

$$\Rightarrow u_t(x,t) = C_2 \sin(\frac{n\pi}{4}x)(\frac{n\pi}{4}C_4)$$

Since $C_2 \neq 0 \Rightarrow C_4 = 0$

$$\Rightarrow u(x,t) = C_2 \sin(\frac{n\pi}{4}x)(C_3 \cos(\frac{n\pi}{4}t) = C_2C_3 \sin(\frac{n\pi}{4}x) \cos(\frac{n\pi}{4}t)$$

Since $u(x,0) = 5\sin(\frac{\pi}{4}x)$

$$\Rightarrow u(x,0) = C_2C_3 \sin(\frac{n\pi}{4}x) = 5\sin(\frac{\pi}{4}x) \Rightarrow C_1C_2 = 5$$

$$\therefore u(x,t) = 5\sin(\frac{n\pi}{4}x)\cos(\frac{n\pi}{4}t)$$

$\square$

**Homework of Wave Equation**

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2} = 0$$

   With boundary condition

$$u(0,t) = 0, u(k^2,t) = 0, u(x,0) = \sin(\frac{\pi}{k^2}x), u_t(x,0) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2} = 0$$

   With boundary condition

$$u(0,t) = 0, u(9,t) = 0, u(x,0) = 2\sin(\frac{\pi}{9}x), u_t(x,0) = 0$$