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# Numerical Analysis

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الرياضيات المتقدمة: المرحلة الثانية

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# Lecture 1

# 1 Introduction to Numerical Analysis

Numerical analysis is the study of algorithms used to solve mathematical problems that are difficult or impossible to solve analytically. These algorithms are employed to provide approximate solutions to problems in various fields such as engineering, physics, economics, and computer science.

The primary objectives of numerical analysis include:

- Developing efficient algorithms.
- Estimating the error of approximate solutions.
- Ensuring the stability of computations.
- Solving equations, both linear and non-linear.
- Differentiation and integration of functions.
- Solving systems of equations.
- Solving differential equations.

## 1.1 Computers and Numerical Analysis

$$Numerical\ Methods + Program\ Computers = Numerical\ Analysis$$

- As you will learn enough about many numerical methods, you will be able to write programs to implement them.
- Programs can be written in any computer language. In this course all programs will be written in Matlab environment.

## 1.2 Types of Errors in Numerical Analysis

Numerical methods are approximate by nature, and errors are introduced at various stages. The most common types of errors are:

- **Round-off Error**: Caused by the limited precision with which computers store real numbers.
- **Truncation Error**: Arises when an infinite process (such as a series expansion) is approximated by a finite number of terms.
- **Absolute Error**: The difference between the exact value and the approximate value.

$$\text{Absolute Error} = |x_{\text{exact}} - x_{\text{approx}}|$$

- **Relative Error**: The absolute error divided by the magnitude of the exact value, giving a measure of the error relative to the exact quantity.

$$\text{Relative Error} = \frac{|x_{\text{exact}} - x_{\text{approx}}|}{|x_{\text{exact}}|}$$

**Illustrative Example: Round-off Error**

Consider the subtraction of two nearly equal numbers using limited precision. Let:

$$x = 1.234567 \quad \text{and} \quad y = 1.234564$$

Their exact difference is:

$$x - y = 0.000003$$

However, if both $x$ and $y$ are stored using only four decimal places, we have:

$$x = 1.2346 \quad \text{and} \quad y = 1.2346$$

Thus, the difference becomes:

$$x - y = 0$$

This demonstrates a loss of precision due to round-off error.

**Illustrative Example: Truncation Error**

Consider the Taylor series expansion of $e^x$ around $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If we approximate $e^x$ by truncating the series after the first two terms, the approximation becomes:

$$e^x \approx 1 + x$$

For small $x$, this approximation may be reasonable, but for larger $x$, the truncation error increases.

For example, when $x = 1$:

$$e^1 = 2.71828 \quad \text{and} \quad 1 + 1 = 2$$

The truncation error is:

$$\text{Truncation Error} = |2.71828 - 2| = 0.71828d$$

# 2 Solution of Non-linear Equations

Non-linear equations are equations of the form:

$$f(x) = 0$$

where $f(x)$ is a non-linear function. In many cases, these equations cannot be solved analytically, and numerical methods are required to approximate the solution. There are various techniques used to solve non-linear equations, such as the Newton-Raphson method, Bisection method, and Secant method.

## 2.1 Newton-Raphson Method

The Newton-Raphson method is an iterative technique for finding approximate solutions to real-valued functions. It's particularly useful for finding roots of equations.

### Method Description

Let $f, f' \in [a, b]$ are continuous function on $[a, b]$. The formula for the Newton-Raphson iteration is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

where:

- $x_n$ is the current approximation,
- $f(x_n)$ is the function evaluated at $x_n$,
- $f'(x_n)$ is the derivative of the function evaluated at $x_n$.

*Remark* 2.1. $\{x_n\}$ be a sequence of approximate roots for $f$, such that:

$$f'(x) \neq 0, \quad |x_n - \alpha| < \epsilon$$

Where, $\alpha$ is the exact root for $f(x) = 0$.

*Remark* 2.2. In order to guarantee that, the iterative process is convergent, the initial root, $x_0$, should be chosen close to the exact root $\alpha$, which means:

$$|x_0 - \alpha| < \epsilon$$

**Steps of Newton-Raphson Algorithm:**

**Step 1** Choose appropriate initial root $x_0 \in [a, b]$.

**Step 2** Let $f(x)$ be the function for which you want to find the root. Compute the derivative $f'(x)$.

**Step 3** Calculate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

**Step 4** Set $n = n + 1$, and continue in the iterative processes,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until we get the stop condition is satisfied: $E_n = |x_{n+1} - x_n| < \epsilon$

**Examples of Newton-Raphson Algorithm**

**Example 2.1.** Use Newton-Raphson algorithm to find the approximate root of the following equation:

$$f(x) = x^2 - 2 = 0$$

and stop when $E_n = |x_{n+1} - x_n| = 0$, where $x_0 = 1.5$. Also, find the iterative error at each step.

*Solution.*

$$f(x) = x^2 - 2, \quad f'(x) = 2x, \quad x_0 = 1.5$$

$$f(x_0) = f(1.5) = (1,5)^2 - 2 = 0.25$$

$$f'(x_0) = f'(1.5) = 2(1.5) = 3$$

**1st iteration**

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
x_1 &= 1.5 - \frac{0.25}{3} \\
x_1 &= 1.4167
\end{aligned}
$$

**2nd iteration**

$$
\begin{aligned}
f(x_1) &= f(1.4167) = (1.4167)^2 - 2 = 0.007 \\
f'(x_1) &= f'(1.4167) = 2(1.4167) = 2.8334 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
x_2 &= 1.4167 - \frac{0.007}{2.8334} \\
x_2 &= 1.4142
\end{aligned}
$$

The iterative error $E_1 = |x_2 - x_1| = |1.4142 - 1.4167| = 0.0025$

**3rd iteration**

$$
\begin{aligned}
f(x_2) &= f(1.4142) = (1.4142)^2 - 2 = 0 \\
f'(x_2) &= f'(1.4142) = 2(1.4142) = 2.8284 \\
x_3 &= 1.4142 - \frac{f(x_2)}{f'(x_2)} \\
x_2 &= 1.4142 - \frac{0}{2.8284} \\
x_2 &= 1.4142 \\
E_2 &= |x_3 - x_2| = |1.4142 - 1.4142| = 0
\end{aligned}
$$

Approximate root of the equation $x^2 - 2 = 0$ using Newton Raphson method is $1.4142$ (After 3 iterations) $\square$

**Example 2.2.** Use Newton-Raphson algorithm to find the approximate root of the following equation:

$$f(x) = \frac{1}{x} + 1 = 0$$

with $x_0 = -0.5$ for three iterative steps (only find $x_1, x_2, x_3$). Also, find the Relative percent error at each step, where $e_n = |\frac{x_{n+1} - x_n}{x_{n+1}}| \times 100\%$

*Solution.*

$$f(x) = \frac{1}{x} + 1, \quad f'(x) = -\frac{1}{x^2}, \quad x_0 = -0.5$$
$$f(x_0) = f(-0.5) = \frac{1}{-0.5} + 1 = -1$$
$$f'(x_0) = f'(-0.5) = -\frac{1}{(-0.5)^2} = -4$$

**1$^{st}$ iteration**

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.5 - \frac{-1}{-4} = -0.75$$

**2$^{nd}$ iteration**

$$f(x_1) = f(-0.75) = \frac{1}{-0.75} + 1 = -0.3333$$
$$f'(x_1) = f'(-0.75) = -\frac{1}{(-0.75)^2} = -1.7778$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.75 - \frac{-0.3333}{-1.7778} = -0.9375$$

Relative percent error $e_1 = |\frac{x_2 - x_1}{x_2}| \times 100\% = |\frac{-0.9375 - (-0.75)}{-0.9375}| \times 100\% = 20\%$

**3rd iteration**

$$
\begin{aligned}
f(x_2) &= f(-0.9375) = \frac{1}{-0.9375} + 1 = -0.0667 \\
f'(x_2) &= f'(-0.9375) = -\frac{1}{(-0.9375)^2} = -1.7778 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = -0.9375 - \frac{-0.06670}{-1.7778} = -0.9750 \\
e_2 &= |\frac{x_3 - x_2}{x_3}| \times 100\% = |\frac{-0.9750 - (-0.9375)}{-0.9750}| \times 100\% = 3.8462\%
\end{aligned}
$$

□

**Homework of Newton-Raphson Algorithm**

1. Use Newton-Raphson algorithm to find the approximate root of the following equation: $f(x) = x \sin x - 1 = 0$, with $x_0 = 1$ for three iterative steps (only find $x_1, x_2, x_3$). Also, find the iterative error.

2. Use Newton-Raphson algorithm to find the approximate root of the following equation: $f(x) = x^2 - 6 = 0$, with $x_0 = 1$ for two iterative steps (only find $x_1, x_2$). Also, find the Relative percent error at each step.

3. Find a root of the equation $x \sin x + \cos x$, for three iterative steps (only find $x_1, x_2, x_3$), where $x_0 = -1$.
   Hint: $f(x) = x \sin x + \cos x,\ f'(x) = x \cos x$

# 3 Interpolation and Approximation

## 3.1 Lagrange Approximation

Lagrange Interpolation is a way of finding the value of any function at any given point when the function is not given. We use other points on the function to get the value of the function at any required point.

If we are given $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, the Lagrange polynomial $P(x)$ is defined as:

$$P(x) = \sum_{i=0}^{n} y_i \ell_i(x) \quad \text{where } \ell_i(x) \text{ are the Lagrange basis polynomials defined by:}$$

$$\ell_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

$$\ell_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_j)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_j)}$$

For example: For 3 data points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, the Lagrange polynomial is:

$$P(x) = y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x)$$

$$P(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

**Examples of the Lagrange Approximation**

**Example 3.1.** Find $f(2.7)$ using Lagrange's Interpolation formula

| $x_i$ | 2 | 2.5 | 3 |
|---|---|---|---|
| $y_i$ | 0.69315 | 0.91629 | 1.09861 |

*Solution.*

$$\ell_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2.5)(x - 3)}{(2 - 2.5)(2 - 3)} = \frac{x^2 - 5.5x + 7.5}{0.5}$$

$$\ell_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 2)(x - 3)}{(2.5 - 2)(2.5 - 3)} = \frac{x^2 - 5x + 6}{-0.25}$$

$$\ell_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 2)(x - 2.5)}{(3 - 2)(3 - 2.5)} = \frac{x^2 - 4.5x + 5}{0.5}$$

$P(x) = y_0\ell_0(x) + y_1\ell_1(x) + y_2\ell_2(x)$

$P(x) = (0.69315)(\frac{x^2 - 5.5x + 7.5}{0.5}) + (0.91629)(\frac{x^2 - 5x + 6}{-0.25}) + (1.09861)(\frac{x^2 - 4.5x + 5}{0.5})$

$P(x) = (1.3863)(x^2 - 5.5x + 7.5) + (-3.66516)(x^2 - 5x + 6) + (2.1972)(x^2 - 4.5x + 5)$

$P(x) = (1.3863x^2 - 7.6247x + 10.3973) + (-3.66516x^2 + 18.3258x - 21.99096)$

$+ (2.1972x^2 - 9.8874x + 10.9861)$

$P(x) = -0.08166x^2 + 0.8137x - 0.60756$

$x = 2.7 \Rightarrow f(2.7) = P(2.7) = -0.08166x(2.7)^2 + 0.8137(2.7) - 0.60756 = 0.9941$ □

**Example 3.2.** Find the Lagrange interpolation polynomial that takes the values $f(0) = 1, f(1) = 1, f(2) = 2, f(4) = 5.$

*Solution.* Given the points: $f(0) = 1, \quad f(1) = 1, \quad f(2) = 2, \quad f(4) = 5$

The Lagrange basis polynomials $\ell_i(x)$ are:

$$\ell_0(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)} = \frac{x^3 - 7x^2 + 14x - 8}{-8}$$

$$\ell_1(x) = \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} = \frac{(x)(x-2)(x-4)}{(1)(-1)(-3)} = \frac{x^3 - 6x^2 + 8x}{3}$$

$$\ell_2(x) = \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} = \frac{(x)(x-1)(x-4)}{(2)(1)(-2)} = \frac{x^3 - 6x^2 + 4x}{-4}$$

$$\ell_3(x) = \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} = \frac{(x)(x-1)(x-2)}{(4)(3)(2)} = \frac{x^3 - 3x^2 + 2x}{24}$$

The Lagrange interpolation polynomial is:

$$
\begin{aligned}
P(x) &= f_0\ell_0(x) + f_1\ell_1(x) + f_2\ell_2(x) + f_3\ell_3(x) \\
P(x) &= (1)(\frac{x^3-7x^2+14x-8}{-8}) + (1)(\frac{x^3-6x^2+8x}{3}) + (2)(\frac{x^3-6x^2+4x}{-4}) \\
&\quad + (5)(\frac{x^3-3x^2+2x}{24}) \\
P(x) &= (-0.125)(x^3-7x^2+14x-8) + (0.3333)(x^3-6x^2+8x) \\
&\quad + (-0.5)(x^3-6x^2+4x) + (0.2083)(x^3-3x^2+2x) \\
P(x) &= (-0.125x^3+0.875x^2-1.75x+1) + (0.3333x^3-2x^2+2.6667x) \\
&\quad + (-0.5x^3+3x^2-2x) + (0.2083x^3-0.625x^2+0.4167x)
\end{aligned}
$$

$\Rightarrow P(x) = -0.0833x^3 + 1.25x^2 - 0.6667x + 1$

H.W: Find $P(3)$ $\square$

**Homework of Lagrange Approximation**

1. Find $f(150)$ using Lagrange's Interpolation formula

| $x_i$ | 300 | 304 |
|---|---|---|
| $y_i$ | 2.4829 | 2.4771 |

2. If $y(1) = 12, y(2) = 15, y(5) = 25,$ and $y(6) = 30.$ Find the four points Lagrange interpolation polynomial that takes some value of function $(y)$ at the given points and estimate the value of $y(4)$ at given points.

3. Fit a cubic through the first four points $y(3.2) = 22.0$, $y(2.7) = 17.8$, $y(1.0) = 14.2$,and $y(5.6) = 51.7$, to find the interpolated value for $x = 3.0$ function $(y)$ at the given points and estimate the value of $y(4)$ at given points.

4. If $f(1.0) = 0.7651977$, $f(1.3) = 0.6200860$, $f(1.6) = 0.4554022$, $f(1.9) = 0.2818186$ and $f(2.2) = 0.1103623$. Use Lagrange polynomial to approximation to $f(1.5)$.

# Lecture 2

# 4 Numerical Integration

Integration is the process of measuring the area under a function plotted on a graph. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods have been developed to find the integral.

Numerical integration refers to techniques used to approximate the value of definite integrals. Instead of solving an integral analytically, these methods estimate the integral using a finite sum of values of the function at specific points. There some types of integration which are Trapezoidal rule, Simpsons 1/3 rule, and Simpsons 3/8 rule.

## 4.1 Trapezoidal Method

The Trapezoidal Method is a numerical integration technique used to approximate the value of a definite integral. It works by dividing the area under the curve into a series of trapezoids rather than rectangles, which often results in a better approximation than a simple Riemann sum.

### Trapezoidal Rule

If you want to approximate the integral of a function $f(x)$ over the interval $[a, b]$, the Trapezoidal Rule can be expressed as:

$$\begin{aligned} I = \int_a^b f(x)\,dx &\approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] \\ &\approx \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)\right] \end{aligned}$$

such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$h = \frac{b-a}{n}$, and $x_i = x_{i-1} + h$ are the points at which the function is evaluated.

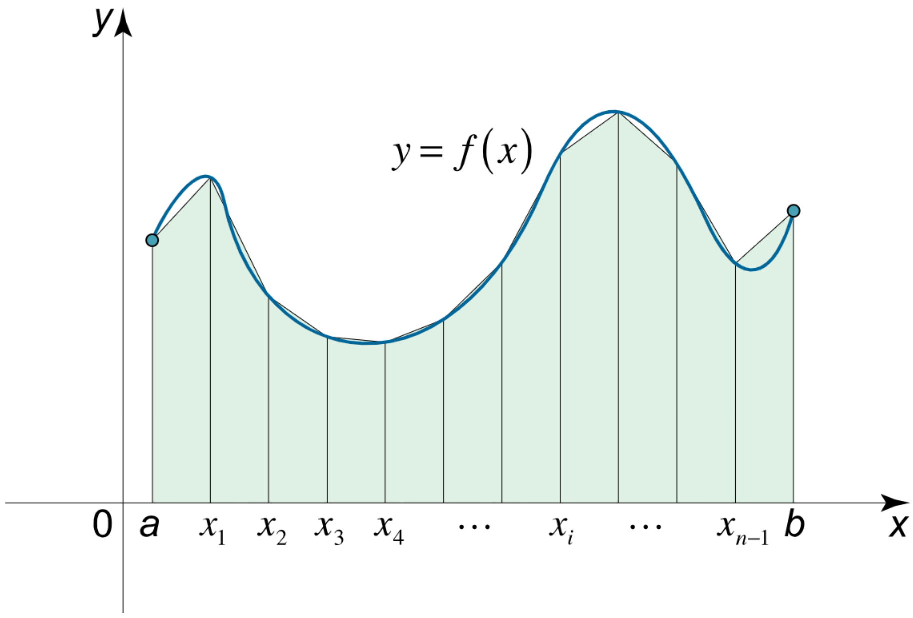


Figure 1: Trapezoidal Method

**Examples of Trapezoidal Rule**

**Example 4.1.** Let us approximate the integral $\int_0^2 e^{x^2}\,dx$ using the Trapezoidal Rule with $n = 5$ intervals.

*Solution.* The interval is $[0, 2]$, and we are dividing it into $n = 5$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 5 \Rightarrow x_0, x_1.x_2, x_3, x_4, x_5$.

$$
\begin{aligned}
x_0 &= a = 0 \\
x_1 &= x_0 + h = 0 + 0.4 = 0.4 \\
x_2 &= x_1 + h = 0.4 + 0.4 = 0.8 \\
x_3 &= x_2 + h = 0.8 + 0.4 = 1.2 \\
x_4 &= x_3 + h = 1.2 + 0.4 = 1.6 \\
x_5 &= x_4 + h = 1.6 + 0.4 = 2 = b
\end{aligned}
$$

**Step 3**: Evaluate the function at these points

$$
\begin{aligned}
f(x_0) &= e^{x_0^2} = f(0) = e^{0^2} = 1 \\
f(x_1) &= e^{x_1^2} = f(0.4) = e^{(0.4)^2} = e^{0.16} \approx 1.1735 \\
f(x_2) &= e^{x_2^2} = f(0.8) = e^{(0.8)^2} = e^{0.64} \approx 1.8965 \\
f(x_3) &= e^{x_3^2} = f(1.2) = e^{(1.2)^2} = e^{1.44} \approx 4.2207 \\
f(x_4) &= e^{x_4^2} = f(1.6) = e^{(1.6)^2} = e^{2.56} \approx 12.9358 \\
f(x_5) &= e^{x_5^2} = f(2) = e^{(2)^2} = e^{4} \approx 54.5982
\end{aligned}
$$

**Step 4**: Apply the Trapezoidal Rule

$$
\begin{aligned}
I = \int_a^b f(x)\,dx = \int_0^2 e^{x^2}\,dx &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right] \\
&\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + f_4) + f_5\right] \\
&\approx \frac{0.4}{2}\left[1 + 2(1.1735 + 1.8965 + 4.2207 + 12.9358) + 54.5982\right] \\
&\approx 0.2\left[1 + 2(20.2265) + 54.5982\right] = 0.2\left[96.0512\right] = 19.2102
\end{aligned}
$$

$$\therefore I \approx 19.2102$$

$\square$

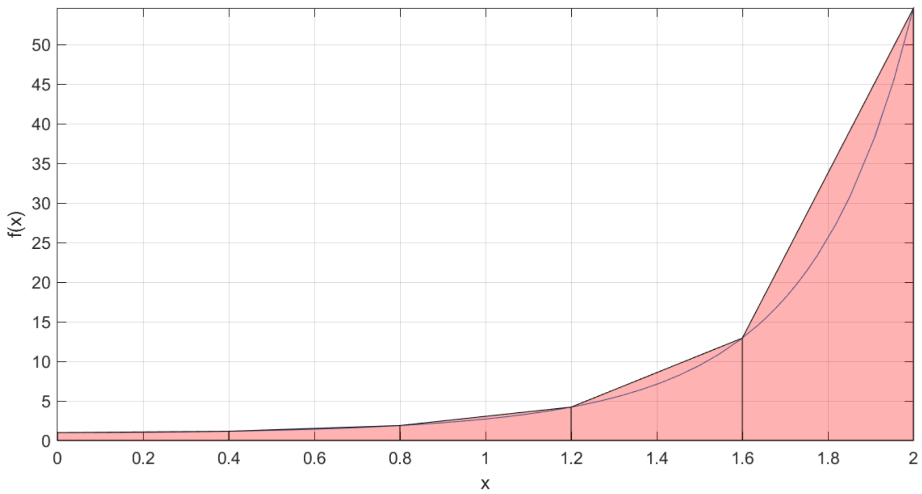


Figure 2: Trapezoidal Method Approximation for $f(x) = e^{x^2}$

**Example 4.2.** Evaluate the integral $\ln(x)$ by trapezoidal rule dividing the interval $[1, 2]$ into four equal parts.

*Solution.* The interval is $[1, 2]$, and we are dividing it into $n = 4$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 4 \Rightarrow x_0, x_1.x_2, x_3, x_4$.

$$x_0 = a = 1, \quad x_1 = x_0 + h = 1 + 0.25 = 1.25$$

$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$

$$x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$

$$x_4 = x_3 + h = 1.75 + 0.25 = 2 = b$$

**Step 3**: Evaluate the function at these points

$$f_0 = \ln(x_0) = f(1) = \ln(1) = 0, \quad f_1 = \ln(x_1) = f(1.25) = \ln(1.25) \approx 0.2231$$

$$f_2 = \ln(x_2) = f(1.5) = \ln(1.5) \approx 0.4055$$

$$f_3 = \ln(x_3) = f(1.75) = \ln(1.75) \approx 0.5596, f_4 = \ln(x_4) = f(2) = \ln(2) \approx 0.6931$$

**Step 4**: Apply the Trapezoidal Rule

$$
\begin{aligned}
I = \int_a^b f(x)\,dx &\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right] \\
&\approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right] \\
&\approx \frac{0.25}{2}\left[0 + 2(0.2231 + 0.4055 + 0.5596) + 0.6931\right] \\
&\approx 0.125\left[0 + 2(1.1882) + 0.6931\right] = 0.125\left[3.0695\right] = 0.3837
\end{aligned}
$$

$$\therefore I \approx 0.3837$$

$\square$

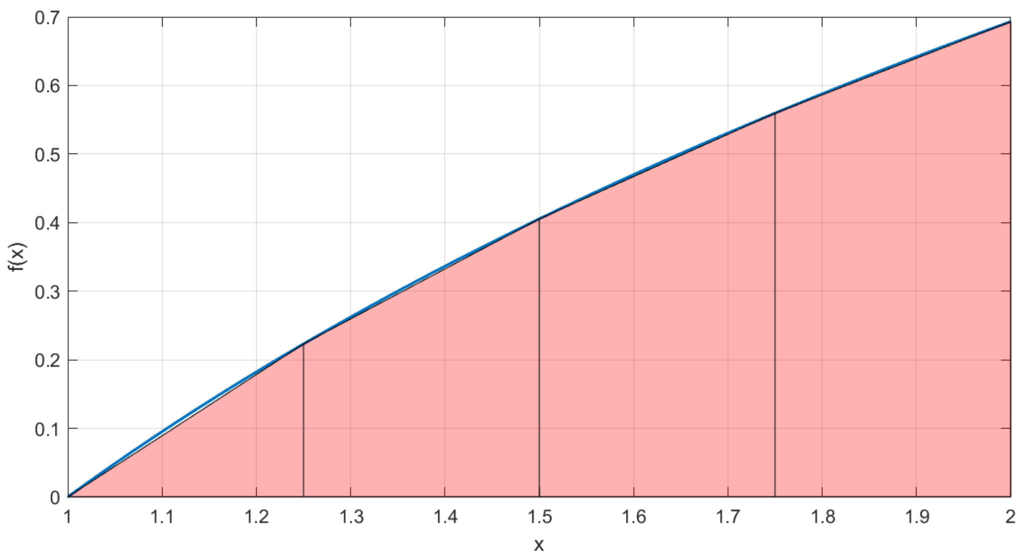


Figure 3: Trapezoidal Method Approximation for $f(x) = \ln(x)$

**Homework of Trapezoidal Rule**

(a) Approximate $\int_0^1 \sin(x^2)dx$, by the trapezoidal rule, with $n = 6$.

(b) Evaluate the integral $\sqrt{x^3 + 1}$ by trapezoidal rule dividing the interval $[0, 3]$ into five equal parts.

## 4.2 Simpsons Method

The Simpsons method is another numerical for approximating the definite integral of a function. It generally provides a more accurate approximation than the Trapezoidal Rule for the same number of intervals.

**Simpsons 1/3 Rule**

If you want to approximate the integral of a function $f(x)$ over the interval $[a, b]$, the Simpsons 1/3 Rule can be expressed as:

$$I = \int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right]$$

such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$h = \frac{b-a}{n}$, and $x_i = x_{i-1} + h$ are the points at which the function is evaluated.

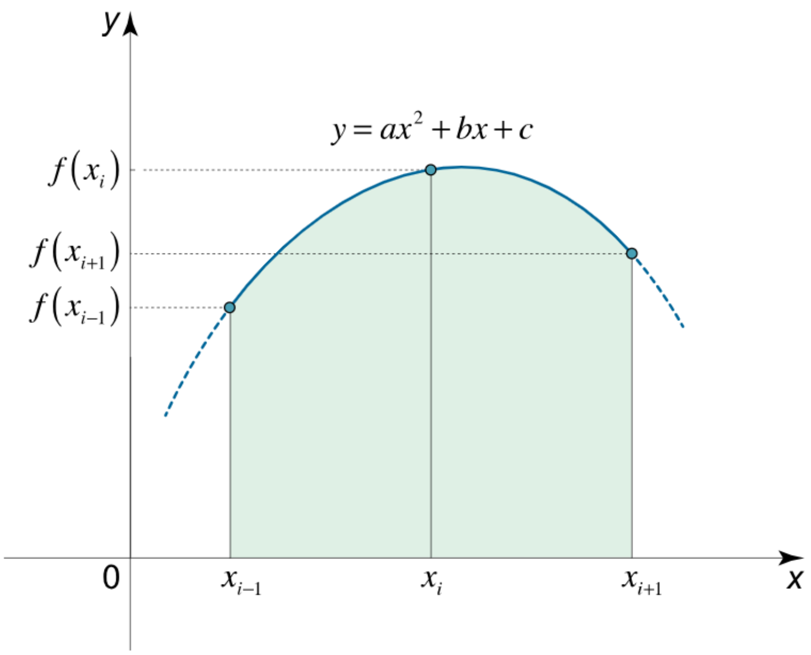


Figure 4: Simpsons 1/3 Method

**Examples of Simpsons 1/3 Rule**

**Example 4.3.** Let us approximate the integral $\int_0^2 e^{x^2}\,dx$ using the Simpsons 1/3 Rule with $n = 5$ intervals.

*Solution.* The interval is $[0, 2]$, and we are dividing it into $n = 5$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 5 \Rightarrow x_0, x_1.x_2, x_3, x_4, x_5$.

$$x_0 = a = 0$$

$$x_1 = x_0 + h = 0 + 0.4 = 0.4$$

$$x_2 = x_1 + h = 0.4 + 0.4 = 0.8$$

$$x_3 = x_2 + h = 0.8 + 0.4 = 1.2$$

$$x_4 = x_3 + h = 1.2 + 0.4 = 1.6$$

$$x_5 = x_4 + h = 1.6 + 0.4 = 2 = b$$

**Step 3**: Evaluate the function at these points

$$f(x_0) = e^{x_0^2} = f(0) = e^{0^2} = 1$$

$$f(x_1) = e^{x_1^2} = f(0.4) = e^{(0.4)^2} = e^{0.16} \approx 1.1735$$

$$f(x_2) = e^{x_2^2} = f(0.8) = e^{(0.8)^2} = e^{0.64} \approx 1.8965$$

$$f(x_3) = e^{x_3^2} = f(1.2) = e^{(1.2)^2} = e^{1.44} \approx 4.2207$$

$$f(x_4) = e^{x_4^2} = f(1.6) = e^{(1.6)^2} = e^{2.56} \approx 12.9358$$

$$f(x_5) = e^{x_5^2} = f(2) = e^{(2)^2} = e^{4} \approx 54.5982$$

**Step 4**: Apply the Simpsons 1/3 Rule

$$\begin{aligned}
I = \int_a^b f(x)\,dx &\approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right] \\
&\approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3) + 2(f_2 + f_4) + f_5\right] \\
&\approx \frac{0.4}{3}\left[1 + 4(1.1735 + 4.2207) + 2(1.8965 + 12.9358) + 54.5982\right] \\
&\approx \frac{0.4}{3}\left[1 + 21.5768 + 29.6646 + 54.5982\right] = \frac{0.4}{3}\left[106.8396\right] \\
\therefore I &\approx 14.2453
\end{aligned}$$

□

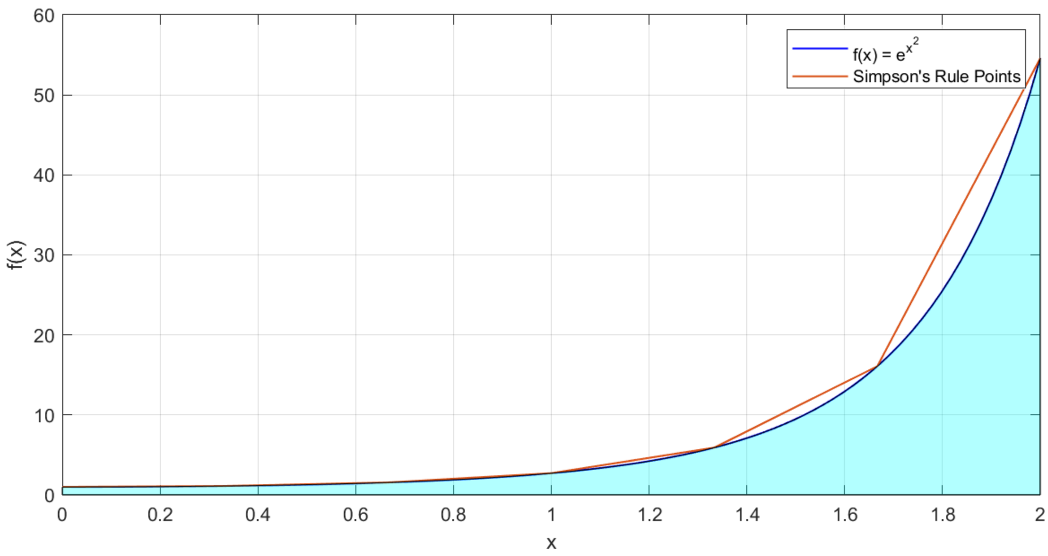


Figure 5: Simpsons 1/3 Method Approximation for $f(x) = e^{x^2}$

**Example 4.4.** Evaluate the integral ln$(x)$ by Simpsons 1/3 rule dividing the interval $[1, 2]$ into four equal parts.

*Solution.* The interval is $[1, 2]$, and we are dividing it into $n = 4$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 4 \Rightarrow x_0, x_1.x_2, x_3, x_4$.

$$x_0 = a = 1, \quad x_1 = x_0 + h = 1 + 0.25 = 1.25$$
$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$
$$x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$
$$x_4 = x_3 + h = 1.75 + 0.25 = 2 = b$$

**Step 3**: Evaluate the function at these points

$$f_0 = \ln(x_0) = f(1) = \ln(1) = 0, \quad f_1 = \ln(x_1) = f(1.25) = \ln(1.25) \approx 0.2231$$
$$f_2 = \ln(x_2) = f(1.5) = \ln(1.5) \approx 0.4055$$
$$f_3 = \ln(x_3) = f(1.75) = \ln(1.75) \approx 0.5596, f_4 = \ln(x_4) = f(2) = \ln(2) \approx 0.6931$$

**Step 4**: Apply the Simpsons 1/3 Rule

$$\begin{aligned} I = \int_a^b f(x)\,dx &\approx \frac{h}{3}\left[f(x_0) + 4\sum_{i\ odd} f(x_i) + 2\sum_{i\ even} f(x_i) + f(x_n)\right] \\ &\approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3) + 2(f_2) + f_4\right] \\ &\approx \frac{0.25}{3}\left[0 + 4(0.2231 + 0.5596) + 2(0.4055) + 0.6931\right] \\ &\approx \frac{0.25}{3}\left[0 + 3.1308 + 0.811 + 0.6931\right] = \frac{0.25}{3}\left[4.6349\right] = 0.3862 \end{aligned}$$

□

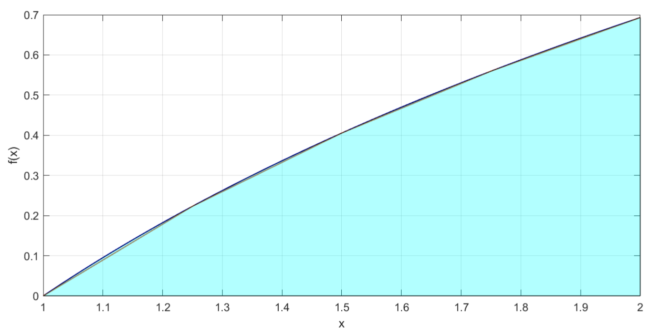


Figure 6: Simpsons 1/3 Method Approximation for $f(x) = \ln(x)$

**Homework of Simpsons 1/3 Rule**

(a) Approximate $\int_0^1 \sin(x^2)dx$, by the Simpsons 1/3 rule, with $n = 6$.

(b) Evaluate the integral $\sqrt{x^3 + 1}$ by Simpsons 1/3 rule dividing the interval $[0, 3]$ into five equal parts.

# Lecture 3

## 4.3 Simpsons 3/8 Method

The Simpsons 3/8 method is another numerical integration used to approximate the definite integral of a function. It is similar to Simpsons 1/3 Rule but uses cubic polynomials instead of quadratic polynomials to approximate the function over subintervals.

$$I = \int_a^b f(x)\, dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i\neq 3k} f(x_i) + 2\sum_{i=3k} f(x_i) + f(x_n)\right]$$

such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$h = \frac{b-a}{n}$, and $x_i = x_{i-1} + h$ are the points at which the function is evaluated.

**Examples of Simpsons 3/8 Rule**

**Example 4.5.** Let us approximate the integral $\int_0^2 e^{x^2}\, dx$ using the Simpsons 3/8 Rule with $n = 5$ intervals.

*Solution.* The interval is $[0, 2]$, and we are dividing it into $n = 5$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-0}{5} = 0.4$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 5 \Rightarrow x_0, x_1.x_2, x_3, x_4, x_5$.

$$x_0 = a = 0$$
$$x_1 = x_0 + h = 0 + 0.4 = 0.4$$
$$x_2 = x_1 + h = 0.4 + 0.4 = 0.8$$
$$x_3 = x_2 + h = 0.8 + 0.4 = 1.2$$
$$x_4 = x_3 + h = 1.2 + 0.4 = 1.6$$
$$x_5 = x_4 + h = 1.6 + 0.4 = 2 = b$$

**Step 3**: Evaluate the function at these points

$$f(x_0) = e^{x_0^2} = f(0) = e^{0^2} = 1$$
$$f(x_1) = e^{x_1^2} = f(0.4) = e^{(0.4)^2} = e^{0.16} \approx 1.1735$$
$$f(x_2) = e^{x_2^2} = f(0.8) = e^{(0.8)^2} = e^{0.64} \approx 1.8965$$
$$f(x_3) = e^{x_3^2} = f(1.2) = e^{(1.2)^2} = e^{1.44} \approx 4.2207$$
$$f(x_4) = e^{x_4^2} = f(1.6) = e^{(1.6)^2} = e^{2.56} \approx 12.9358$$
$$f(x_5) = e^{x_5^2} = f(2) = e^{(2)^2} = e^{4} \approx 54.5982$$

**Step 4**: Apply the Simpsons 3/8 Rule

$$\begin{aligned}
I = \int_a^b f(x)\,dx &\approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i\neq 3k} f(x_i) + 2\sum_{i=3k} f(x_i) + f(x_n)\right] \\
&\approx \frac{3h}{8}\left[f_0 + 3(f_1 + f_2 + f_4) + 2(f_3) + f_5\right] \\
&\approx \frac{3 \times 0.4}{8}\left[1 + 3(1.1735 + 1.8965 + 12.9358) + 2(4.2207) + 54.5982\right] \\
&\approx \frac{1.2}{8}\left[1 + 48.0174 + 8.4414 + 54.5982\right] = 0.15\,[112.057]
\end{aligned}$$

$$\therefore I \approx 16.8086$$

□

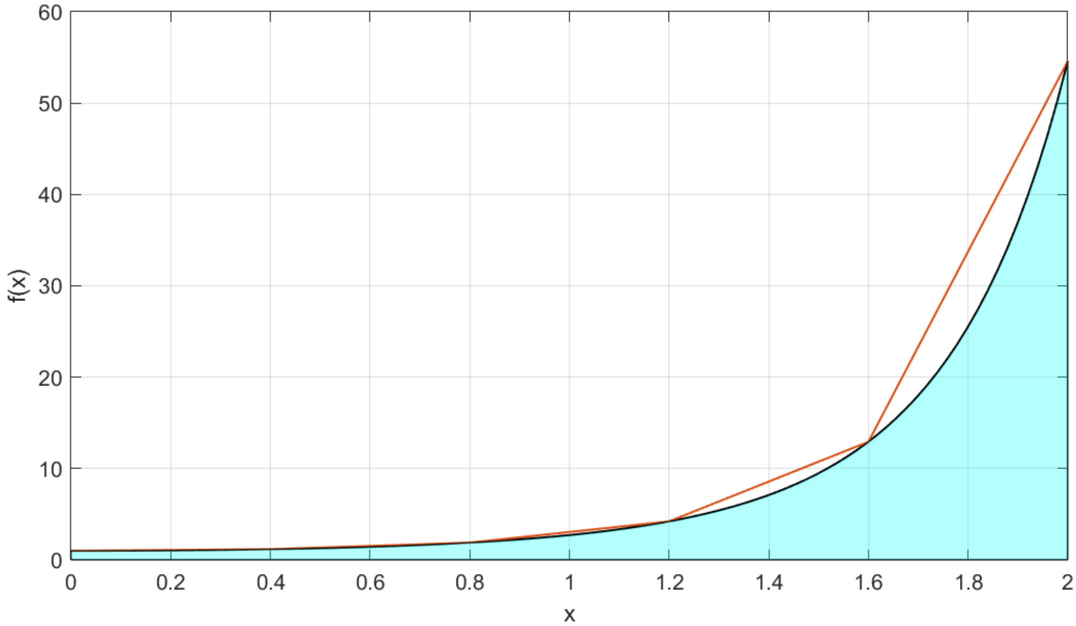


Figure 7: Simpsons 3/8 Method Approximation for $f(x) = e^{x^2}$

**Example 4.6.** Evaluate the integral ln$(x)$ by Simpsons 3/8 rule dividing the interval $[1, 2]$ into four equal parts.

*Solution.* The interval is $[1, 2]$, and we are dividing it into $n = 4$ subintervals.

**Step 1**: Calculate $h$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$$

**Step 2**: Determine the evaluation points $x_i$, since $n = 4 \Rightarrow x_0, x_1.x_2, x_3, x_4$.

$$x_0 = a = 1, \quad x_1 = x_0 + h = 1 + 0.25 = 1.25$$

$$x_2 = x_1 + h = 1.25 + 0.25 = 1.5 \quad x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$

$$x_4 = x_3 + h = 1.75 + 0.25 = 2 = b$$

**Step 3**: Evaluate the function at these points

$f_0 = \ln(x_0) = f(1) = \ln(1) = 0, \quad f_1 = \ln(x_1) = f(1.25) = \ln(1.25) \approx 0.2231$

$f_2 = \ln(x_2) = f(1.5) = \ln(1.5) \approx 0.4055$

$f_3 = \ln(x_3) = f(1.75) = \ln(1.75) \approx 0.5596, f_4 = \ln(x_4) = f(2) = \ln(2) \approx 0.6931$

**Step 4**: Apply the Simpsons 3/8 Rule

$$\begin{aligned}
I = \int_a^b f(x)\,dx &\approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i\neq 3k} f(x_i) + 2\sum_{i=3k} f(x_i) + f(x_n)\right] \\
&\approx \frac{3h}{8}\left[f_0 + 3(f_1 + f_2) + 2(f_3) + f_4\right] \\
&\approx \frac{3\times 0.25}{8}\left[0 + 3(0.2231 + 0.4055) + 2(0.5596) + 0.6931\right] \\
&\approx \frac{0.75}{8}\left[0 + 1.8858 + 1.1192 + 0.6931\right] = 0.3467
\end{aligned}$$

□

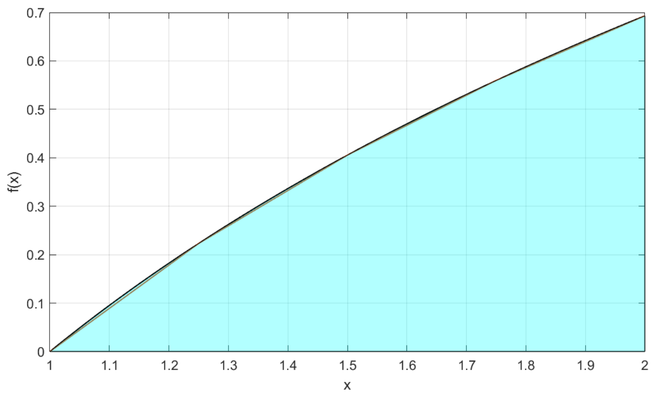


Figure 8: Simpsons 3/8 Method Approximation for $f(x) = \ln(x)$

**Homework of Simpsons 3/8 Rule**

(a) Approximate $\int_0^1 \sin(x^2)dx$, by the Simpsons 3/8 rule, with $n = 6$.

(b) Evaluate the integral $\sqrt{x^3 + 1}$ by Simpsons 3/8 rule dividing the interval $[0, 3]$ into five equal parts.

(c) Find $\int_0^1 4x^3 dx$, with $n = 4$ by:

i. The exact value

ii. The Trapezoidal rule

iii. The Simpsons 1/3 rule

iv. The Simpsons 3/8 rule

v. Compare all solutions

| Methods | Solutions |
|---|---|
| Exact | |
| Trapezoidal | |
| Simpsons 1/3 | |
| Simpsons 3/8 | |

## Lecture 4

# 5 Numerical Differentiation

Numerical differentiation is a method used to approximate the derivative of a function when an analytical solution is difficult or impossible to obtain. It's an important technique in numerical analysis and computational mathematics. Here's an overview of numerical differentiation:

1. **Basic Concept:** The derivative of a function $f(x)$ at a point $x$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   Numerical differentiation approximates this limit using finite differences.

2. **Finite Differences:** Finite differences are a numerical method used to approximate derivatives and solve difference equations. Here are some common forms of finite differences and their definitions

- **Forward Difference:** $f'(x) \approx \frac{f(x+h)-f(x)}{h}$. This method uses the function value at $x$ and $(x + h)$ .
- **Backward Difference:** $f'(x) \approx \frac{f(x)-f(x-h)}{h}$. This method uses the function value at $x$ and $(x - h)$ .
- **Central Difference:** $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$. This method is generally more accurate as it uses the function values at both ($x + h$) and ($x - h$).

3. **Higher-Order Derivatives:** For more accuracy, higher-order methods can be used. These methods involve more points and higher-order terms in the Taylor series expansion.

# 6 Solutions of Ordinary Differential Equation

Numerical differentiation is often used in the context of solving ordinary differential equations (ODEs) when analytical solutions are difficult or impossible to obtain.

Solving ODEs numerically involves finding the function $y(t)$ that satisfies the differential equation $\frac{dy}{dt} = f(t, y)$ given an initial condition $y(t_0) = y_0$. Here are some common numerical methods for solving ODEs:

1. Euler Method
2. Modified Euler Method
3. Rung Kutta Method
4. Rung Kutta-Merson Method

## 6.1 Euler Method

The Euler method is a numerical technique for solving ordinary differential equations (ODEs) with a given initial value. It is defined as follows:

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The Euler method updates the solution using the formula:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where $h$ is the step size, $t_n$ is the current time, and $y_n$ is the current value of the solution.

**The Step by Step Method**

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

**Step 1.** We find $t_i$

$$\begin{aligned} t_1 &= t_0 + h \\ t_2 &= t_1 + h \\ &\vdots \\ t_{n+1} &= t_n + h \end{aligned}$$

**Step 2.** We find $y_i$

$$\begin{aligned} y_1 &= y_0 + h \cdot f(t_0, y_0) \\ y_2 &= y_1 + h \cdot f(t_1, y_1) \\ &\vdots \\ y_{n+1} &= y_n + h \cdot f(t_n, y_n) \end{aligned}$$

**Example 6.1.** Use Euler's method to solve the $D.E.$ and find $y_3$

$$\frac{dy}{dt} = \frac{t^2 - y}{2}, \quad \text{with } t_0 = 0, y_0 = 4, h = 0.05$$

*Sol.* $f(t, y) = \frac{t^2-y}{2}$

**Step 1.** We find $t_i$

$$t_0 = 0$$
$$t_{n+1} = t_n + h$$
$$t_1 = t_0 + h = 0 + 0.05 = 0.05$$
$$t_2 = t_1 + h = 0.05 + 0.05 = 0.1$$

**Step 2.** We find $y_i$

$$y_0 = 4$$
$$y_{n+1} = y_n + h \cdot f(t_n, y_n)$$
$$f(t_0, y_0) = \frac{t_0^2 - y_0}{2} = \frac{0^2 - 4}{2} = -2$$
$$y_1 = y_0 + h \cdot f(t_0, y_0) = 4 + 0.05(-2) = 4 + (-0.1) = 3.9$$
$$f(t_1, y_1) = \frac{t_1^2 - y_1}{2} = \frac{(0.05)^2 - 3.9}{2} = -1.9488$$
$$y_2 = y_1 + h \cdot f(t_1, y_1) = 3.9 + 0.05(-1.9488) = 3.9 + (-0.0974) = 3.8026$$
$$f(t_2, y_2) = \frac{t_2^2 - y_2}{2} = \frac{(0.1)^2 - 3.8026}{2} = -1.8963$$
$$y_3 = y_2 + h \cdot f(t_1, y_1) = 3.8026 + 0.05(-1.8963) = 3.8026 + (-0.0948) = 3.7077$$

□

**Example 6.2.** Use Euler's method to solve the $D.E.$ and find $y_4$

$$\frac{dy}{dt} + 2y = 1.3e^{-t}, \quad \text{with } t_0 = 0, y_0 = 0.5, h = 1$$

*Sol.* $\frac{dy}{dt} + 2y = 1.3e^{-t} \Rightarrow \frac{dy}{dt} = 1.3e^{-t} - 2y \Rightarrow f(t, y) = 1.3e^{-t} - 2y$

**Step 1.** We find $t_i$

$$\begin{aligned} t_0 &= 0 \\ t_{n+1} &= t_n + h \\ t_1 &= t_0 + h = 0 + 1 = 1 \\ t_2 &= t_1 + h = 1 + 1 = 2 \\ t_3 &= t_3 + h = 2 + 1 = 3 \end{aligned}$$

**Step 2.** We find $y_i$

$$\begin{aligned} y_0 &= 0.5 \\ y_{n+1} &= y_n + h \cdot f(t_n, y_n) \\ f(t_0, y_0) &= 1.3e^{-t_0} - 2y_0 = 1.3e^{-0} - 2(0.5) = 0.3 \\ y_1 &= y_0 + h \cdot f(t_0, y_0) = 0.5 + (1)(0.3) = 0.5 + 0.3 = 0.8 \\ f(t_1, y_1) &= 1.3e^{-t_1} - 2y_1 = 1.3e^{-1} - 2(0.8) = -1.1218 \\ y_2 &= y_1 + h \cdot f(t_0, y_0) = 0.8 + (-1.1218) = -0.3218 \\ f(t_2, y_2) &= 1.3e^{-t_2} - 2y_2 = 1.3e^{-2} - 2(-0.3218) = 0.8194 \\ y_3 &= y_2 + h \cdot f(t_2, y_2) = -0.3218 + 0.8194 = 0.4977 \\ f(t_3, y_3) &= 1.3e^{-t_3} - 2y_3 = 1.3e^{-3} - 2(0.4977) = -0.9307 \\ y_4 &= y_3 + h \cdot f(t_2, y_2) = 0.4977 + (-0.9307) = -0.433 \end{aligned}$$

□

**Homework of Euler Method**

1. Use Euler's method to solve $\frac{dy}{dt} = t - y^2$ and find $y_5$, when $t_0 = 0, y_0 = 1, h = 0.1$.

2. Use Euler's method to solve the $D.E.$ and find $y_4$

$$\frac{dy}{dx} + y = -xy^2, \quad \text{with } x_0 = 0, y_0 = 1, h = 0.1$$

## 6.2 Modified Euler Method

The Modified Euler Method gives from modified the value of $(y_n + 1)$ at point $(x_n + 1)$ by gives the new value $(y_n + 1)$.

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The method involves the following steps:

1. **Predictor Step (Euler Method)**:

$$y^*_{n+1} = y_n + hf(t_n, y_n)$$

2. **Corrector Step (Modified Euler's Method)**:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right]$$

**The Step by Step Method**

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

**Step 1.** Using Euler's method $y^*_{n+1} = y_n + hf(t_n, y_n)$

**Step 2.** Using $y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right]$

**Example 6.3.** Use Modified Euler's method to solve the $D.E.$ and find $y_2$

$$\frac{dy}{dt} = \frac{t^2 - y}{2}, \quad \text{with } t_0 = 0, y_0 = 4, h = 0.05$$

*Sol.* $f(t, y) = \frac{t^2-y}{2}$

**Step 1.** We find $t_i$ and $y^*_i$

$$t_0 = 0 \qquad t_{n+1} = t_n + h$$

$$t_1 = t_0 + h = 0 + 0.05 = 0.05 \qquad t_2 = t_1 + h = 0.05 + 0.05 = 0.1$$

$$y_0 = y^*_0 = 4$$

$$y^*_{n+1} = y^*_n + h \cdot f(t_n, y^*_n)$$

$$f(t_0, y^*_0) = \frac{t_0^2 - y^*_0}{2} = \frac{0^2 - 4}{2} = -2$$

$$y^*_1 = y^*_0 + h \cdot f(t_0, y^*_0) = 4 + 0.05(-2) = 4 + (-0.1) = 3.9$$

$$f(t_1, y^*_1) = \frac{t_1^2 - y^*_1}{2} = \frac{(0.05)^2 - 3.9}{2} = -1.9488$$

$$y^*_2 = y^*_1 + h \cdot f(t_1, y^*_1) = 3.9 + 0.05(-1.9488) = 3.9 + (-0.0974) = 3.8026$$

$$f(t_2, y^*_2) = \frac{t_2^2 - y^*_2}{2} = \frac{(0.1)^2 - 3.8026}{2} = -1.8963$$

**Step 2.** We find $y_i$

$$
\begin{aligned}
y_0 &= y_0^* = 4 \\
f(t_0, y_0^*) &= f(t_0, y_0) = -2, \;\; f(t_1, y_1^*) = -1.9488, \;\; f(t_2, y_2^*) = -1.8963 \\
y_{n+1} &= y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}^*)\right] \\
y_1 &= y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, y_1^*)\right] = 4 + \frac{0.05}{2}\left[-2 + (-1.9488)\right] = 3.9013 \\
f(t_1, y_1) &= \frac{t_1^2 - y_1}{2} = \frac{(0.05)^2 - 3.9013}{2} = -1.9494 \\
y_2 &= y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, y_2^*)\right] = 3.9013 + \frac{0.05}{2}\left[-1.9494 + (-1.8963)\right] \\
&= 3.9013 + \frac{0.05}{2}\left[-3.8457\right] = 3.9013 + (-0.0961) = 3.8052
\end{aligned}
$$

□

**Homework of Modified Euler Method**

Use Modified Euler method to solve the $D.E.$ and find $y_2$

$$
\frac{dy}{dt} + 2y = 1.3e^{-t}, \quad \text{with } t_0 = 0, y_0 = 0.5, h = 1
$$

# Lecture 5

## 6.3 Rung Kutta Method

The Runge-Kutta Method of order 4 (RK4) is one of the most commonly used methods for solving ordinary differential equations (ODEs). It provides a good balance between accuracy and computational efficiency.

Given the initial value problem:

$$
\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0
$$

The method uses the following intermediate slopes:

$$k_1 = h \cdot f(t_n, y_n),$$
$$k_2 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$
$$k_4 = h \cdot f(t_n + h, y_n + k_3).$$

The solution is updated using:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

**The Step by Step Method**

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

**Step 1.** We find $t_i$

$$t_1 = t_0 + h$$
$$t_2 = t_1 + h$$
$$t_3 = t_2 + h$$
$$\vdots$$
$$t_{n+1} = t_n + h$$

**Step 2.** We find $k_i$

$$\begin{aligned}
k_1 &= h \cdot f(t_n, y_n), \\
k_2 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\
k_3 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\
k_4 &= h \cdot f(t_n + h, y_n + k_3).
\end{aligned}$$

**Step 3.** We find $y_i$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

**Example 6.4.** Use Rung Kutta method to solve the $D.E.$ and find $y_1$

$$\frac{dy}{dt} = y - t, \quad \text{with } t_0 = 0, y_0 = 2, h = 0.1$$

*Sol.* $f(t, y) = y - t$

**Step 1.** We find $t_i$: $t_0 = 0$

**Step 2** We find $k_i$ when $n = 0$

$$\begin{aligned}
&f(t, y) = y - t \\
k_1 &= h \cdot f(t_0, y_0) = h \cdot f(0, 2) = 0.1(2 - 0) = 0.2 \\
k_2 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right) = 0.1 \cdot f(0.05, 2.1) \\
&= 0.1(2.1 - 0.05) = 0.205 \\
k_3 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 2 + \frac{0.205}{2}\right) = 0.1 \cdot f(0.05, 2.1025) \\
&= 0.1(2.1025 - 0.05) = 0.2053 \\
k_4 &= h \cdot f(t_0 + h, y_0 + k_3) = 0.1 \cdot f(0 + 0.1, 2 + 0.2053) = 0.1 \cdot f(0.1, 2.2053) \\
&= 0.1(2.2053 - 0.1) = 0.2105
\end{aligned}$$

**Step 3.** We find $y_1$

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) = 2 + \frac{1}{6}(0.2 + 2(0.205 + 0.2053) + 0.2105) \\
&= 2.20693
\end{aligned}
$$

□

**Example 6.5.** Use Rung Kutta method to solve the $D.E.$ and find $y_2$

$$\frac{dy}{dt} = ty, \quad \text{with } t_0 = 0.1, y_0 = 4, h = 0.9$$

*Sol.* $f(t, y) = ty$

**Step 1.** We find $t_i$

$$t_0 = 0, \quad t_1 = t_0 + h = 0.1 + 0.9 = 1$$

**Step 2** We find $k_i$ when $n = 0$

$$
\begin{aligned}
& f(t, y) = ty \\
k_1 &= h \cdot f(0.1, 4) = h \cdot f(0.1, 4) = 0.9(0.1 \times 4) = 0.36 \\
k_2 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.9 \cdot f\left(0.1 + \frac{0.9}{2}, 4 + \frac{0.36}{2}\right) \\
&= 0.9 \cdot f(0.55, 4.18) = 0.9(0.55 \times 4.18) = 2.0691 \\
k_3 &= h \cdot f\left(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.9 \cdot f\left(0.1 + \frac{0.9}{2}, 4 + \frac{2.0691}{2}\right) \\
&= 0.9 \cdot f(0.55, 5.0346) = 0.9(0.55 \times 5.0346) = 2.4921 \\
k_4 &= h \cdot f(t_0 + h, y_0 + k_3) = 0.9 \cdot f(0.1 + 0.9, 4 + 2.4921) \\
&= 0.9 \cdot f(1, 6.4921) = 0.9(1 \times 6.4921) = 5.8429
\end{aligned}
$$

**Step 3.** We find $y_1$

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\ y_1 &= y_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) = 4 + \frac{1}{6}(0.36 + 2(2.0691 + 2.4921) + 5.8429) \\ &= 6.5542 \end{aligned}$$

Again **(Steps 2,3)** taking $t_1 = 1, y_1 = 6.5542$ to find $k_i$ and $y_2$.

**Step 2** We find $k_i$ when $n = 1$

$$\begin{aligned} &f(t, y) = ty \\ &k_1 = h \cdot f(t_1, y_1) = h \cdot f(1, 6.5542) = 0.9(1 \times 6.5542) = 5.8988 \\ &k_2 = h \cdot f\left(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.9 \cdot f\left(1 + \frac{0.9}{2}, 6.5542 + \frac{5.8988}{2}\right) \\ &\quad = 0.9 \cdot f(1.45, 9.5036) = 0.9(1.45 \times 9.5036) = 12.4022 \\ &k_3 = h \cdot f\left(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.9 \cdot f\left(1 + \frac{0.9}{2}, 6.5542 + \frac{12.4022}{2}\right) \\ &\quad = 0.9 \cdot f(1.45, 12.7553) = 0.9(1.45 \times 12.7553) = 16.6457 \\ &k_4 = h \cdot f(t_1 + h, y_1 + k_3) = 0.9 \cdot f(1 + 0.9, 6.5542 + 16.6457) \\ &\quad = 0.9 \cdot f(1.9, 23.1999) = 0.9(1.9 \times 23.1999) = 39.6719 \end{aligned}$$

**Step 3.** We find $y_1$

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\ y_2 &= y_2 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \\ y_2 &= 6.5542 + \frac{1}{6}(5.8988 + 2(12.4022 + 16.6457) + 39.6719) = 23.832 \end{aligned}$$

□

**Homework of Rung Kutta Method**

Use Rung Kutta method to solve the $D.E.$ and find $y_3$

$$\frac{dy}{dt} = t + y, \quad \text{with } t_0 = 0, y_0 = 0, h = 0.2$$

## 6.4 Runge-Kutta-Merson Method

The Runge-Kutta-Merson method is a specific type of Runge-Kutta method that is particularly well-suited for solving ordinary differential equations (ODEs) with a high degree of accuracy. It is a fifth-order method, which means that it is accurate to the fifth order of accuracy.

Given the initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

The method uses the following intermediate slopes:

$$\begin{aligned}
k_1 &= h \cdot f(t_n, y_n), \\
k_2 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right), \\
k_3 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}\right), \\
k_4 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right), \\
k_5 &= h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4)
\end{aligned}$$

The solution is updated using:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_4 + k_5)$$

The error is estimated as:

$$\text{Error} = \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5)$$

**The Step by Step Method**

Given an initial value problem:

$$\frac{dy}{dt} = f(t, y), \quad t_0 = a, y_0 \quad \text{with } h \quad \text{where } y(t_0) = y_0,$$

**Step 1.** We find $t_i$

$$\begin{aligned} t_1 &= t_0 + h \\ t_2 &= t_1 + h \\ t_3 &= t_2 + h \\ &\vdots \\ t_{n+1} &= t_n + h \end{aligned}$$

**Step 2.** We find $k_i$

$$\begin{aligned} k_1 &= h \cdot f(t_n, y_n), \\ k_2 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right), \\ k_3 &= h \cdot f\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{6} + \frac{k_2}{6}\right), \\ k_4 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right), \\ k_5 &= h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4) \end{aligned}$$

**Step 3.** We find $y_i$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_4 + k_5)$$

**Step 4.** The error is estimated

$$\text{Error} = \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5)$$

**Example 6.6.** Use Runge-Kutta-Merson method to solve the $D.E.$ and find $y_1$

$$\frac{dy}{dt} = t + y, \quad \text{with } t_0 = 0, y_0 = 1, h = 0.1$$

*Sol.* $f(t, y) = t + y$

**Step 1.** We find $t_i$

$$t_0 = 0$$

**Step 2** We find $k_i$ when $n = 0$

$$\begin{aligned}
& f(t,y) = t + y \\
k_1 &= h \cdot f(t_0, y_0) = h \cdot f(0, 1) = 0.1(0 + 1) = 0.1 \\
k_2 &= h \cdot f\left(t_0 + \frac{h}{3}, y_0 + \frac{k_1}{3}\right) = h \cdot f\left(0 + \frac{0.1}{3}, 1 + \frac{0.1}{3}\right) \\
&= h \cdot f\left(0.0333, 1.0333\right) = 0.1(0.0333 + 1.0333) = 0.1067 \\
k_3 &= h \cdot f\left(t_0 + \frac{h}{3}, y_0 + \frac{k_1}{6} + \frac{k_2}{6}\right) = h \cdot f\left(0 + \frac{0.1}{3}, 1 + \frac{0.1}{6} + \frac{0.1067}{6}\right) \\
&= h \cdot f\left(0.0333, 1.0345\right) = 0.1(0.0333 + 1.0345) = 0.1068 \\
k_4 &= h \cdot f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{8} + \frac{3k_3}{8}\right) = h \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{8} + \frac{3(0.1068)}{8}\right) \\
&= h \cdot f\left(0.05, 1.0526\right) = 0.1(0.05 + 1.0526) = 0.1103 \\
k_5 &= h \cdot f(t_n + h, y_n + \frac{k_1}{2} - \frac{3k_3}{2} + 2k_4) \\
&= h \cdot f(0 + 0.1, 1 + \frac{0.1}{2} - \frac{3(0.1068)}{2} + 2(0.1103)) = h \cdot f(0.1, 1.1104) \\
&= 0.1 \cdot (0.1 + 1.1104) = 0.1210
\end{aligned}$$

**Step 3.** We find $y_1$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_4 + k_5)$$
$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_4 + k_5) = 1 + \frac{1}{6}(0.1 + 4(0.1103) + 0.1210) = 1.1104$$

**Step 4.** The error is estimated

$$\text{Error} = \frac{1}{30}(2k_2 - 9k_3 + 8k_4 - k_5)$$
$$\text{Error} = \frac{1}{30}(2(0.1) - 9(0.1068) + 8(0.1103) - 0.1210) = 6.6667 \times 10^{-6}$$

□

**Homework of Runge-Kutta-Merson Method**

Use Runge-Kutta-Merson method to solve the $D.E.$ and find $y_2$

$$\frac{dy}{dt} = ty, \quad \text{with } t_0 = 0.1, y_0 = 4, h = 0.9$$

## Lecture 6

# 7 Partial Differential Equations

Partial Differential Equation (PDE)is an equation that involves multiple independent variables, an unknown function that depends on these variables, and partial derivatives of the unknown function. PDEs are used to formulate problems involving functions of several variables and are especially important in describing physical phenomena such as heat conduction, wave propagation, fluid flow, and electromagnetism.

A general form of a PDE can be written as:

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1^2}, \ldots\right) = 0$$

where $u = u(x_1, x_2, \ldots, x_n)$ is the unknown function, and $F$ represents a relationship between $u$ and its partial derivatives.

## 7.1 Classification of Partial Differential Equations

Partial differential equations are classified according to many things. Classification is an important concept because the general theory and methods of solution usually apply only to a given class of equations.

### 7.1.1 Order of the Partial Differential Equations

The order of a PDE is the order of the highest partial derivative in the equation.

For example:

1. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \; or \; u_t + cu_x = 0$ (First-order)
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; or \; u_{xx} + u_{yy} = 0$ (Second-order)
3. $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$ (First-order)
4. $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0 \; or \; u_{xxx} + u_{xy} + u_y = 0$ (Third-order)

### 7.1.2 Degree of Partial Differential Equations

The degree of a partial differential equation is the degree of the Highest order partial derivative occurring in the equation.

For example:

1. $u_{xx} + u_y^2 + u = 0$ (First-degree)
2. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (First-degree)

3. $u_t + uu_x = 0$ (Second-degree)

4. $\left(\frac{\partial^4 u}{\partial x^4}\right)^3 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial u}{\partial x} = 0$ (Third-degree)

5. $e^{\frac{\partial u}{\partial x}} + \sin\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$ (Undefined)

**Homework: Determine the Order and Degree of the PDEs**

For each of the following PDEs, determine the order and degree:

1. $\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} = 0$

2. $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 1$

3. $\frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$

4. $\frac{\partial^3 u}{\partial x^3} + u^2\frac{\partial u}{\partial t} = \sin(x)$

5. $u_{xx} + 2u_{yy} + 3u_{xy} + 4 = 0$

### 7.1.3 Number of Variables of Partial Differential Equations

1. $u_t = u_x$ (Two variables: $x$ and $t$)

2. $u_t = u_{tt} + \frac{1}{r}u_r + u_\theta$ (Three variables: $r$, $\theta$, and $t$)

**Homework: Identify the Number of Variables**

For each of the following PDEs, identify the number of independent variables:

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

2. $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \alpha u$

3. $\frac{\partial u}{\partial t} + \beta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial y^2} = f(x, y, t)$

### 7.1.4 Linearity Partial Differential Equations

A PDE is linear if the unknown function and its partial derivatives appear to the first power and are not multiplied together. More precisely, a second order linear equation in two variables is an equation of the form:

$$a(x, y, z, \dots) * u_{xx} + b(x, y, z, \dots) * u_{xy} + \dots + f(x, y, z, \dots) * u = g(x, y, z, \dots)$$

where $u$ is the unknown function, $x, y, z, \dots$ are the independent variables, and $a, b, c, \dots$ and $f, g$ are functions of the independent variables only.

1. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ (Linear)
2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (Linear)
3. $u_{tt} = e^{-t} u_{xx} = sin(x)$ (Linear)
4. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ (Non-Linear)
5. $u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0$ (Non-Linear)

### 7.1.5 Homogeneity of Partial Differential Equations

A PDE is classified as either **Homogeneous** or **Non-Homogeneous** based on the presence of a non-zero term that does not involve the unknown function or its derivatives. A PDE is said to be **Homogeneous** if all terms involve either the unknown function or its derivatives, and there is no standalone term (i.e., the right-hand side of the equation is zero).

In contrast, a PDE is called **Non-Homogeneous** if there is a non-zero term on the right-hand side of the equation that is independent of the unknown function.

1. $\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = 0$ (homogeneous)
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (homogeneous)

3. $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = g(x,t)$ (non-homogeneous)

4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (non-homogeneous)

**Homework: Linearity and Homogeneity**

Classify the following PDEs as linear or nonlinear, and as homogeneous or non-homogeneous:

1. $\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

2. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$

3. $\frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial y^2} + g(x,y) = 0$

4. $\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta u = 0$

## 7.2 Solution of Partial Differential Equations

Partial Differential Equations (PDEs) arise in various fields such as physics, engineering, and finance. The solutions to these equations can describe physical phenomena like heat conduction, wave propagation, and fluid dynamics. In the section presents several methods to solve PDEs along with examples.

### 7.2.1 Solution of Partial Differential Equations by Direct Integration

Direct integration is a technique used to solve certain types of first-degree Partial Differential Equations (PDEs) where the equation can be rearranged to allow for straightforward integration with respect to one variable.

**Example 7.1.** Find the solution of the following PDE

$$\frac{\partial^2 z}{\partial x^2} = xy$$

*Sol.*

$$\frac{\partial^2 z}{\partial x^2} = xy \tag{1}$$

Integrate Equation (1) w.r.t. $x$.

$$\frac{\partial z}{\partial x} = \int xy \, dx + f(y) = (\frac{x^2}{2})y + f(y)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. $x$.

$$\begin{aligned} z &= \int \frac{x^2 y}{2} + f(y) \, dx + g(y) \\ &= \frac{y}{2} \int x^2 + f(y) \, dx + g(y) \\ &= \frac{y}{2} \int x^2 \, dx + \int f(y) \, dx + g(y) \\ &= \frac{y}{2}(\frac{x^3}{3}) + x \cdot f(y) + g(y) \end{aligned}$$

$$\therefore z = \frac{x^3 y}{6} + x \cdot f(y) + g(y)$$

Where $f(y), g(y)$ is arbitrary parametric. □

**Example 7.2.** Find the solution of the following PDE

$$\frac{\partial^2 z}{\partial xy} = 0$$

*Sol.*

$$\frac{\partial^2 z}{\partial xy} = 0 \tag{1}$$

Integrate Equation (1) w.r.t. $x$.

$$\frac{\partial z}{\partial y} = \int 0 \, dx + f(y) = f(y)$$

$$\therefore \frac{\partial z}{\partial y} = f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. $y$.

$$z = \int f(y)\, dy + g(x)$$

Where $f(y), g(y)$ is arbitrary parametric. □

**Example 7.3.** Find the solution of the following PDE

$$\frac{\partial^2 u}{\partial xy} = 6x + 12y^2$$

With boundary condition, $u(1, y) = y^2 - 2y$, $u(x, 2) = 5x - 5$

*Sol.*

$$\frac{\partial^2 u}{\partial xy} = 6x + 12y^2 \tag{1}$$

Integrate Equation (1) w.r.t. $x$.

$$\frac{\partial u}{\partial y} = \int 6x + 12y^2\, dx + f(y) = \frac{6x^2}{2} + 12y^2 x + f(y)$$

$$\therefore \frac{\partial u}{\partial y} = 3x^2 + 12y^2 x + f(y) \tag{2}$$

Now, Integrate Equation (2) w.r.t. $y$.

$$\begin{aligned} u &= \int 3x^2 + 12y^2 x + f(y)\, dy + g(x) \\ &= 3x^2 y + \frac{12y^3}{3} x + \int f(y)\, dy + g(x) \\ &= 3x^2 y + 4y^3 x + h(y) + g(x) \quad \text{where } h(y) = \int f(y)\, dy \end{aligned}$$

$$u(x,y) = 3x^2y + 4y^3x + h(y) + g(x)$$

$$u(1,y) = 3(1)^2y + 4y^3(1) + h(y) + g(1) = y^2 - 2y$$

$$= 3y + 4y^3 + h(y) + g(1) = y^2 - 2y$$

$$h(y) = -4y^3 + y^2 - 5y - g(1)$$

$$\therefore u(x,y) = 3x^2y + 4y^3x + (-4y^3 + y^2 - 5y - g(1)) + g(x)$$

$$u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - g(1) + g(x)$$

$$u(x,2) = 3x^2(2) + 4(2)^3x - 4(2)^3 + (2)^2 - 5(2) - g(1) + g(x) = 5x - 5$$

$$= 6x^2 + 32x - 32 + 4 - 10 - g(1) + g(x) = 5x - 5$$

$$= 6x^2 + 32x - 38 - g(1) + g(x) = 5x - 5$$

$$g(x) = -6x^2 - 27x + 33 + g(1)$$

$$u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - g(1) + (-6x^2 - 27x + 33 + g(1))$$

$$\therefore u(x,y) = 3x^2y + 4y^3x - 4y^3 + y^2 - 5y - 6x^2 - 27x + 33$$

□

**Homework: Solving Partial Differential Equations by Direct Integration**

Solve the following PDEs by direct integration:

1. $\frac{\partial u}{\partial x} = y^2$
2. $\frac{\partial u}{\partial t} = 3x^2$, with the boundary condition $u(x,0) = x^3$.
3. $\frac{\partial u}{\partial t} = -4x$, with the boundary condition $u(x,0) = e^x$.

# Lecture 7

## 7.3 Formation of PDE by Eliminating Arbitrary Constant

A PDE may formed by a eliminating arbitrary constants or arbitrary function from a given relation and other relation obtained by differentiating partially the given relation.

*Remark* 7.1. Suppose the following relation:

1. $\frac{\partial z}{\partial x} = z_x = p$

2. $\frac{\partial z}{\partial y} = z_y = q$

**Example 7.4.** Form a Partial Differential Equations from the following equation:

$$z = (x-a)^2 + (y-b)^2 \tag{1}$$

*Sol.*

$$z_x = 2(x-a) \Rightarrow (x-a) = \frac{z_x}{2} \Rightarrow -a = \frac{z_x}{2} - x \Rightarrow a = x - \frac{z_x}{2}$$
$$z_y = 2(y-b) \Rightarrow (y-b) = \frac{z_y}{2} \Rightarrow -b = \frac{z_y}{2} - y \Rightarrow b = y - \frac{z_y}{2}$$

Eq. (1) become

$$\begin{aligned} z &= (x - (x - \frac{z_x}{2}))^2 + (y - (y - \frac{z_y}{2}))^2 \\ &= (-\frac{z_x}{2})^2 + (-\frac{z_y}{2})^2 \\ &= \frac{z_x^2}{4} + \frac{z_y^2}{4} \Rightarrow 4z = z_x^2 + z_y^2 = p^2 + q^2 \end{aligned}$$

$$\therefore 4z = p^2 + q^2$$

□

**Example 7.5.** Form a Partial Differential Equations from the following equation:

$$z = f(x^2 + y^2) \tag{1}$$

*Sol.*

$$z_x = 2x \cdot f'(x^2+y^2) \Rightarrow f'(x^2+y^2) = \frac{z_x}{2x} \tag{2}$$

$$z_y = 2y \cdot f'(x^2+y^2) \Rightarrow f'(x^2+y^2) = \frac{z_y}{2y} \tag{3}$$

Sub. Eq. (2) in Eq (3)

$$z_y = 2y\frac{z_x}{2x} \Rightarrow \frac{z_y}{z_x} = \frac{y}{x}$$

$$\therefore \frac{q}{p} = \frac{y}{x}$$

□

**Example 7.6.** Form a Partial Differential Equations from the following equation:

$$z = ax + by + a^2 + b^2 \tag{1}$$

*Sol.*

$$z_x = a$$

$$z_y = b$$

Eq. (1) become

$$z = z_x x + z_y y + (z_x)^2 + (z_y)^2$$

$$\therefore z = px + qy + p^2 + q^2$$

□

**Example 7.7.** Form a Partial Differential Equations from the following equation:

$$v = f(x-ct) + g(x+ct) \tag{1}$$

*Sol.*

$$v_x = f'(x-ct) + g'(x+ct)$$

$$v_t = -cf'(x-ct) + cg'(x+ct)$$

$$v_{xx} = f''(x-ct) + g''(x+ct)$$

$$v_{tt} = c^2 f''(x-ct) + c^2 g''(x+ct)$$

$$v_{tt} = c^2(f''(x-ct) + g''(x+ct))$$

$$v_{tt} = c^2(v_{xx})$$

$$\therefore v_{tt} = c^2 v_{xx}$$

□

**Homework of Formation of PDE by Eliminating Arbitrary Constant**

Form a Partial Differential Equations from the following:

1. $z = ax + by + a^2 + b^2$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$
3. $z = f(\frac{y}{x})$
4. $f(x - at) + g(x + at)$

## 7.4 Method of Separation of Variables

Although there are several methods that can be tried to find particular solutions of a linear PDE, in the method of separation of variables we seek to find a particular solution of the form of a product of a function of $x$ and a function of $y$,

$$u(x, y) = X(x)Y(y).$$

With this assumption, it is sometimes possible to reduce a linear PDE in two variables to two ODEs. To this end we observe that

$$\frac{\partial u}{\partial x} = X'Y \quad \frac{\partial u}{\partial y} = XY' \quad \frac{\partial^2 u}{\partial x^2} = X''Y \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

where the primes denote ordinary differentiation.

$$X' = \frac{dX}{dx} \quad Y' = \frac{dY}{dy}$$

**Example 7.8.** Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0 \tag{1}$$

With boundary condition

$$u(0, y) = 4e^{-2y} - 3e^{-6y} \tag{2}$$

To solve Eq. (1) suppose $u(x, y) = XY$.

Then

$$\frac{\partial u}{\partial x} = X'Y \quad \frac{\partial u}{\partial y} = XY'$$

$$X' = \frac{dX}{dx} \quad Y' = \frac{dY}{dy}$$

Put in Eq. (1)

$$YX' + 3XY' = 0$$

$$\frac{X'}{3X} = -\frac{Y'}{Y}$$

Now let

$$\frac{X'}{3X} = -\frac{Y'}{Y} = c \quad c \text{ constant}$$

$$\frac{X'}{3X} = c, \quad -\frac{Y'}{Y} = c$$

$$\Rightarrow X' = 3cX, \quad Y' = -cY, \Rightarrow X = a_1 e^{3cx}, \quad Y = a_2 e^{-cy},$$

$$\Rightarrow u(x,y) = XY = a_1 e^{3cx} a_2 e^{-cy} = a_1 a_2 e^{3cx-cy} = Be^{c(3x-y)}, \quad B = a_1 a_2$$

Now let

$$u(x,y) = u_1 + u_2 = b_1 e^{c_1(3x-y)} + b_2 e^{c_2(3x-y)}$$

$$\Rightarrow u(0,y) = b_1 e^{c_1(-y)} + b_2 e^{c_2(-y)} = 4e^{-2y} - 3e^{-6y}$$

$$\Rightarrow b_1 = 4, b_2 = -3, c_1 = 2, c_2 = 6$$

$$\therefore u(x,y) = 4e^{2(3x-y)} - 3e^{6(3x-y)}$$

**Homework of Method of Separation of Variables**

1. Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} = 0$$

With boundary condition

$$u(0,y) = 4e^{-6y} - 5e^{-y}$$

2. Solve the following Partial Differential Equation with boundary condition

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

With boundary condition

$$u(0,y) = 2e^{-6y} + e^{-2y}$$

# Lecture 8

## 7.5 Heat Equation

The heat equation is a fundamental partial differential equation that describes how heat (or temperature) evolves over time in a given region. It is widely used in physics, engineering, and mathematics to model heat conduction.

### The One-Dimensional Heat Equation

In its simplest form, the one-dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ is the temperature distribution function, representing the temperature at position $x$ and time $t$.
- $k$ is the thermal diffusivity constant, which depends on the material properties.
- $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with respect to time.
- $\frac{\partial^2 u}{\partial x^2}$ represents the spatial second derivative of temperature, indicating how temperature varies along the spatial axis.

### Solution of the Heat Equation

The method of separation of variables is commonly used to solve the heat equation, leading to solutions of the form:

$$u(x,t) = X(x)T(t)$$

Substituting $u(x,t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

**Example 7.9** (Heat Equation)**.** Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0,t) = 0, u(10,t) = 0 \tag{2}$$

*Sol.* To solve Eq. (1) suppose $u(x,t) = XT$.

Then

$$\frac{\partial u}{\partial t} = XT' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Since $u(0,t) = 0, u(10,t) = 0 \Rightarrow X(0) = 0, X(10) = 0$

Put in Eq. (1)

$$XT' = 2X''T$$
$$\frac{T'}{2T} = \frac{X''}{X}$$

Now let

$$\frac{T'}{2T} = \frac{X''}{X} = b, \quad b \text{ constant}$$
$$\frac{T'}{2T} = b \Rightarrow T' = 2bT$$
$$\frac{X''}{X} = b \Rightarrow X'' = 2bX \Rightarrow X'' - 2bX = 0$$

Now let $b = \lambda^2$

when $\lambda^2 \geq 0$ trivial solution. Then $b = -\lambda^2 < 0$

$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Since $X(0) = 0$,

$$\Rightarrow X(0) = A\cos(\lambda(0)) + B\sin(\lambda(0)) = A \Rightarrow A = 0,$$

$$X(10) = A\cos(\lambda(10)) + B\sin(\lambda(10)) = 0 \Rightarrow X(10) = B\sin(\lambda(10))$$

$$\therefore \sin(\lambda(10)) = 0$$

Since $\sin(\lambda(10)) = 0 \Rightarrow 10\lambda = n\pi, n = 0, 1, 2, \ldots$

$$\Rightarrow \lambda = \frac{n\pi}{10}$$

$$\therefore x(x) = B_n \sin(\frac{n\pi}{10}x)$$

Since $T' = 2bT \Rightarrow T' = e^{2bt} \Rightarrow T' = e^{-2\lambda^2 t} \Rightarrow T' = e^{-2(\frac{n\pi}{10})^2 t}$

$$\Rightarrow T = C_n e^{-2(\frac{n\pi}{10})^2 t}$$

$$\therefore u(x,t) = XT = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{10}x)e^{-2(\frac{n\pi}{10})^2 t}, A_n = B_n C_n$$

□

**Homework of Heat Equation**

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} = 0, \quad k \text{ constant}$$

With boundary condition

$$u(0,t) = 0, u(L,t) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial u}{\partial t} = 6\frac{\partial^2 u}{\partial x^2} = 0$$

With boundary condition

$$u(0,t) = 0, u(30,t) = 0$$

## 7.6 Wave Equation

The wave equation is a second-order partial differential equation that describes the propagation of waves, such as sound waves, light waves, or water waves, in a given medium. It is a fundamental equation in physics and engineering, with applications in fields such as acoustics, electromagnetism, and fluid dynamics.

**The Wave Equation in One Dimension**

In one-dimensional space, the wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where:

- $u(x,t)$ represents the displacement of the wave at position $x$ and time $t$,
- $c$ is the speed of wave propagation in the medium.
- Second Partial Derivative with Respect to Time ($\frac{\partial^2 u}{\partial t^2}$):
  - This term represents the **acceleration** of the wave function $u(x,t)$ with respect to time at any given position $x$.
  - It indicates how the wave's displacement changes over time, capturing the oscillatory nature of waves.

- Second Partial Derivative with Respect to Space ($\frac{\partial^2 u}{\partial x^2}$):
  - This term measures the **curvature** of the wave function $u(x, t)$ with respect to spatial dimensions.
  - Physically, it represents how the wave's displacement changes along the spatial dimension(s), indicating how the wave bends or curves at any given point.

**Solution of the Wave Equation**

The method of separation of variables is commonly used to solve the wave equation, leading to solutions of the form:

$$u(x, t) = X(x)T(t)$$

Substituting $u(x, t) = X(x)T(t)$ into the heat equation and separating the variables results in two ordinary differential equations, one for $X(x)$ and one for $T(t)$, which can be solved individually under the given initial and boundary conditions.

**Example 7.10.** Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

With boundary condition

$$u(0, t) = 0, u(4, t) = 0, u(x, 0) = 5\sin(\frac{\pi}{4}x), u_t(x, 0) = 0 \tag{2}$$

*Sol.* To solve Eq. (1) suppose $u(x, t) = XT$.

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \frac{\partial^2 u}{\partial X^2} = X''T$$

Put in Eq. (1)

$$XT'' = 4X''T \Rightarrow \frac{T''}{4T} = \frac{X''}{X}$$
$$\frac{T''}{4T} = \frac{X''}{X} = -p^2, \quad p^2 \text{ constant}$$

Now let

$$\frac{T''}{4T} = -p^2 \Rightarrow T'' + 4p^2T = 0$$
$$\frac{X''}{X} = -p^2 \Rightarrow X'' + 4p^2X = 0$$

$$\Rightarrow X(x) = C_1\cos(px) + C_2\sin(px)$$

$$T(t) = C_3\cos(pt) + C_4\sin(pt)$$

$$\Rightarrow u(x,t) = (C_1\cos(px) + C_2\sin(px))(C_3\cos(pt) + C_4\sin(pt))$$

Since $u(0,t)$,

$$\Rightarrow u(0,t) = C_1(C_3\cos(pt) + C_4\sin(pt)) = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow u(x,t) = C_2\sin(px)(C_3\cos(pt) + C_4\sin(pt))$$

Since $u(0,4)$,

$$\Rightarrow u(4,t) = C_2\sin(4p)(C_3\cos(pt) + C_4\sin(pt)) = 0$$

Since $C_2 \neq 0 \Rightarrow \sin(4p) = 0 \Rightarrow 4p = n\pi \Rightarrow p = \frac{n\pi}{4}, n = 0, 1, \cdots$

$$\Rightarrow u(x,t) = C_2\sin(\frac{n\pi}{4}x)(C_3\cos(\frac{n\pi}{4}t) + C_4\sin(\frac{n\pi}{4}t))$$

$$u_t(x,t) = C_2\sin(\frac{n\pi}{4}x)(-\frac{n\pi}{4}C_3\sin(\frac{n\pi}{4}t) + \frac{n\pi}{4}C_4\cos(\frac{n\pi}{4}t))$$

Since $u_t(x,0) = 0$

$$\Rightarrow u_t(x,t) = C_2\sin(\frac{n\pi}{4}x)(\frac{n\pi}{4}C_4)$$

Since $C_2 \neq 0 \Rightarrow C_4 = 0$

$$\Rightarrow u(x,t) = C_2\sin(\frac{n\pi}{4}x)(C_3\cos(\frac{n\pi}{4}t) = C_2C_3\sin(\frac{n\pi}{4}x)\cos(\frac{n\pi}{4}t)$$

Since $u(x,0) = 5\sin(\frac{\pi}{4}x)$

$$\Rightarrow u(x,0) = C_2C_3\sin(\frac{n\pi}{4}x) = 5\sin(\frac{\pi}{4}x) \Rightarrow C_1C_2 = 5$$

$$\therefore u(x,t) = 5\sin(\frac{n\pi}{4}x)\cos(\frac{n\pi}{4}t)$$

□

**Homework of Wave Equation**

1. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = k^2\frac{\partial^2 u}{\partial x^2} = 0$$

   With boundary condition

$$u(0,t) = 0, u(k^2,t) = 0, u(x,0) = \sin(\frac{\pi}{k^2}x), u_t(x,0) = 0$$

2. Find the solution of following equation by using partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} = 0$$

   With boundary condition

$$u(0,t) = 0, u(9,t) = 0, u(x,0) = 2\sin(\frac{\pi}{9}x), u_t(x,0) = 0$$

# Lecture 9

## 8 Solution of a System of Linear Equations

A system of linear equations is a set of equations where each equation is linear. Consider the following system of $m$ equations with $n$ variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

where $a_{ij}$ are the coefficients, $x_j$ are the variables, and $b_i$ are the constants. In matrix form, the system can be expressed as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

There are several methods to solve this system:

- **Direct Methods:** Gauss Elimination Method
- **Iterative Methods:** Gauss-Seidel
- **Solution by Cramer's Rule**
- **Solution by Matrix Inversion**

## 8.1 Gauss Elimination Method

The Gauss Elimination Method is a systematic technique for solving systems of linear equations. It transforms the system's augmented matrix into an upper triangular form using a series of row operations, making it easier to solve through back-substitution.

**Example 8.1.** Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 2x + 3y = 5 \\ 4x + y = 6 \end{cases}$$

*Sol.* Put the system in the following matrix form

$$\begin{bmatrix} 2 & 3 & \vert & 5 \\ 4 & 1 & \vert & 6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$R_1 \Rightarrow \frac{R_1}{2}$

$$\begin{bmatrix} \frac{2}{2} & \frac{3}{2} & \vert & \frac{5}{2} \\ 4 & 1 & \vert & 6 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \vert & \frac{5}{2} \\ 4 & 1 & \vert & 6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$R_2 \Rightarrow R_2 - 4R_1$

$$\begin{bmatrix} 1 & \frac{3}{2} & \vert & \frac{5}{2} \\ 4 - 4(1) & 1 - 4(\frac{3}{2}) & \vert & 6 - 4(\frac{5}{2}) \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \vert & \frac{5}{2} \\ 0 & -5 & \vert & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$R_2 \Rightarrow \frac{R_2}{-5}$

$$\begin{bmatrix} 1 & \frac{3}{2} & \vert & \frac{5}{2} \\ 0 & \frac{-5}{-5} & \vert & \frac{-4}{-5} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \vert & \frac{5}{2} \\ 0 & 1 & \vert & \frac{4}{5} \end{bmatrix}$$

$$\Rightarrow y = \frac{4}{5} \Rightarrow x + \frac{3}{2}y = \frac{5}{2} \quad \Rightarrow x + \frac{3}{2}(\frac{4}{5}) = \frac{5}{2} \Rightarrow x + \frac{12}{10} = \frac{5}{2} \Rightarrow x = \frac{13}{10}$$

□

**Example 8.2.** Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 3x - y + 2z = 12 \\ 3x + 2y + 3z = 11 \\ 2x - 2y - z = 2 \end{cases}$$

*Sol.* Put the system in the following matrix form

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 3 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 \Rightarrow R_2 - R_1$ and $R_3 \Rightarrow 3R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 3-3 & 2-(-1) & 3-2 & 11-12 \\ 3(2)-2(3) & -2(3)-2(-1) & -1(3)-2(2) & 2(3)-2(12) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & -4 & -7 & -18 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \Rightarrow 3R_3 + 4R_1$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & -4(3)+4(3) & -7(3)+4 & -18(3)-4 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & -17 & -58 \end{array}\right]$$

$$\Rightarrow -17z = -58 \Rightarrow z = \frac{58}{17}$$
$$3y + z = -1 \Rightarrow 3y + \frac{58}{17} \Rightarrow 3y = -1 - \frac{58}{17} = -\frac{75}{17}$$
$$\Rightarrow y = -\frac{75}{17}(\frac{1}{3}) \Rightarrow y = -\frac{25}{17}$$
$$3x - y + 2z = 12 \Rightarrow 3x - (-\frac{25}{17}) + 2(\frac{58}{17}) = 12$$
$$3x + \frac{25}{17} + \frac{116}{17} = 12 \Rightarrow 3x = 12 - \frac{141}{17} = \frac{63}{17}$$
$$\Rightarrow x = \frac{63}{17}(\frac{1}{3}) = \frac{21}{17}$$

□

**Homework of Gauss Elimination Method**

1. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} 3x - 3y = 2 \\ -7x + 2y = 0 \end{cases}$$

2. Solve the system of linear equations by Gauss Elimination Method

$$\begin{cases} x + 2y - 4z = 4 \\ 5x - 3y - 7z = 6 \\ 3x - 4y + 3z = 1 \end{cases}$$

## 8.2 Gauss Siedle Methods

The Gauss-Seidel Method is an iterative technique for solving systems of linear equations, typically used when the coefficient matrix is large and sparse. It uses the most recently updated values for each variable in each iteration, which can to faster convergence. Consider a system of linear equations in matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

**Iteration Formula**

The Gauss-Seidel Method updates each variable $x_i$ as:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \dots, n$$

where $x_i^{(k+1)}$ is the updated value of $x_i$ at the $(k+1)$-th iteration.

Consider the following system of linear equations:

$$\begin{cases} a_x x + a_y y + a_z z = d_1, \\ b_x x + b_y y + b_z z = d_2, \\ c_x x + c_y y + c_z z = d_3 \end{cases}$$

We can rewrite the equations in terms of $x$, $y$, and $z$ for the Gauss-Seidel method:

$$x^{(k+1)} = \frac{1}{a_x} \left( d_1 - a_y y^{(k)} - a_z z^{(k)} \right),$$

$$y^{(k+1)} = \frac{1}{b_y} \left( d_2 - b_x x^{(k+1)} - b_z z^{(k)} \right),$$

$$z^{(k+1)} = \frac{1}{c_z} \left( d_3 - c_x x^{(k+1)} - c_y y^{(k+1)} \right).$$

Starting with initial guesses $x^{(0)} = 0$, $y^{(0)} = 0$, and $z^{(0)} = 0$, we apply the above formulas iteratively until the values converge.

where $a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z$ are constants.

**Example 8.3.** Solving systems of linear equations using Gauss Seidel method

$$\begin{cases} 5x - 2y + z = 4, \\ x + 4y - 2z = 3, \\ x + 4y + 4z = 17. \end{cases}$$

*Sol.*

$$x = \frac{1}{5}(4 + 2y - z)$$

$$y = \frac{1}{4}(3 - x + 2z)$$

$$z = \frac{1}{4}(17 - x - 4y)$$

$$x^{(k+1)} = \frac{1}{5}\left(4 + 2y^{(k)} - z^{(k)}\right),$$

$$y^{(k+1)} = \frac{1}{4}\left(3 - x^{(k+1)} + 2z^{(k)}\right),$$

$$z^{(k+1)} = \frac{1}{4}\left(17 - x^{(k+1)} - 4y^{(k+1)}\right).$$

We start with an initial guess: $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$.

Calculate $x^{(1)}$, $y^{(1)}$, $z^{(1)}$:

$$x^{(1)} = \frac{1}{5}(4 + 2 \times 0 - 0) = \frac{4}{5} = 0.8$$

$$y^{(1)} = \frac{1}{4}(3 - 0.8 + 2 \times 0) = \frac{2.2}{4} = 0.55$$

$$z^{(1)} = \frac{1}{4}(17 - 0.8 - 4 \times 0.55) = \frac{17 - 0.8 - 2.2}{4} = \frac{14}{4} = 3.5$$

After the first iteration, we have $x^{(1)} = 0.8$, $y^{(1)} = 0.55$, and $z^{(1)} = 3.5$.

Calculate $x^{(2)}$, $y^{(2)}$, $z^{(2)}$:

$$x^{(2)} = \frac{1}{5}(4 + 2 \times 0.55 - 3.5) = \frac{4 + 1.1 - 3.5}{5} = \frac{1.6}{5} = 0.32$$

$$y^{(2)} = \frac{1}{4}(3 - 0.32 + 2 \times 3.5) = \frac{3 - 0.32 + 7}{4} = \frac{9.68}{4} = 2.42$$

$$z^{(2)} = \frac{1}{4}(17 - 0.32 - 4 \times 2.42) = \frac{17 - 0.32 - 9.68}{4} = \frac{7}{4} = 1.75$$

$\square$

**Homework of Gauss Siedle Method**

1. Solving systems of linear equations $(x^{(3)}, y^{(3)}, z^{(3)})$ using Gauss Seidel method

$$\begin{cases} 2x - y - 3z = 1, \\ 5x + 2y - 6z = 5, \\ 3x - y - 4z = 7. \end{cases}$$

2. Solving systems of linear equations $(x^{(3)}, y^{(3)}, z^{(3)})$ using Gauss Seidel method

$$\begin{cases} 2x - y + z = 1, \\ 3x - 2yz = 0, \\ 5x + y + 2z = 9. \end{cases}$$

# Lecture 10

## 8.3 Cramer's Rule

Cramer's Rule is a mathematical theorem used to solve systems of linear equations with as many equations as unknowns, provided the determinant of the coefficient matrix is non-zero.

## General Form

Consider a system of linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

We can represent this system in matrix form as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

For Cramer's Rule to apply, we need det($\mathbf{A}$) $\neq 0$. We then solve for $x$, $y$, and $z$ by computing determinants of matrices where each column of **A** is replaced by **b** one at a time:

1. Solution for $x$:

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})}$$

where $\mathbf{A}_x$ is the matrix obtained by replacing the first column of **A** with **b**:

$$\mathbf{A}_x = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

2. Solution for $y$:

$$y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})}$$

where $\mathbf{A}_y$ is the matrix obtained by replacing the second column of **A** with **b**:

$$\mathbf{A}_y = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}.$$

3. Solution for $z$:

$$z = \frac{\det(\mathbf{A}_z)}{\det(\mathbf{A})}$$

where $\mathbf{A}_z$ is the matrix obtained by replacing the third column of **A** with **b**:

$$\mathbf{A}_z = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

**Example 8.4.** Solve the system of linear equations using Cramer's rule

$$\begin{cases} x - 2y = 3, \\ 4x + 5y = 12. \end{cases}$$

*Sol.* This system can be written in matrix form as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

The solution can be found using Cramer's rule:

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

First, calculate the determinant of $\mathbf{A}$:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} = 1 \cdot 5 - (-2) \cdot 4 = 5 + 8 = 13$$

Next, calculate the determinants of $\mathbf{A}_x$ and $\mathbf{A}_y$:

$$\mathbf{A}_x = \begin{bmatrix} 3 & -2 \\ 12 & 5 \end{bmatrix} \Rightarrow \det(\mathbf{A}_x) = \begin{vmatrix} 3 & -2 \\ 12 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 12 = 15 + 24 = 39$$

$$\mathbf{A}_y = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix} \Rightarrow \det(\mathbf{A}_y) = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix} = 1 \cdot 12 - 3 \cdot 4 = 12 - 12 = 0$$

Therefore, the solutions are:

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})} = \frac{39}{13} = 3$$

$$y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})} = \frac{0}{13} = 0$$

Thus, the solution to the system is:

$$x = 3, \quad y = 0$$

□

**Example 8.5.** Solve the system of linear equations using Cramer's rule

$$\begin{cases} 3x - 6y + 7z = 3, \\ 4x - 5z = 3, \\ 5x - 8y + 6z = -4. \end{cases}$$

*Sol.*

$$A = \begin{bmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{bmatrix}$$

The determinant of $A$ is:

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} - (-6) \cdot \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} + 7 \cdot \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix}$$

$$= 3(0 \cdot 6 - (-5) \cdot (-8)) + 6(4 \cdot 6 - (-5) \cdot 5) + 7(4 \cdot (-8) - 0 \cdot 5)$$

$$= 3(0 - 40) + 6(24 + 25) + 7(-32 - 0)$$

$$= 3(-40) + 6(49) + 7(-32)$$

$$= -120 + 294 - 224$$

$$= -50$$

Now, we need to find the determinants of the matrices obtained by replacing each column of $A$ with the constant matrix. Let's denote these matrices as $A_x$, $A_y$, and $A_z$.

$$A_x = \begin{bmatrix} 3 & -6 & 7 \\ 3 & 0 & -5 \\ -4 & -8 & 6 \end{bmatrix}$$

$$A_y = \begin{bmatrix} 3 & 3 & 7 \\ 4 & 3 & -5 \\ 5 & -4 & 6 \end{bmatrix}$$

$$A_z = \begin{bmatrix} 3 & -6 & 7 \\ 4 & 0 & 3 \\ 5 & -8 & -4 \end{bmatrix}$$

The determinants of these matrices are:

$$\det(\mathbf{A}_x) = 3 \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} - (-6) \begin{vmatrix} 3 & -5 \\ -4 & 6 \end{vmatrix} + 7 \begin{vmatrix} 3 & 0 \\ -4 & -8 \end{vmatrix}$$

$$= 3(0 \cdot 6 - (-5) \cdot (-8)) + 6(3 \cdot 6 - (-5) \cdot (-4)) + 7(3 \cdot (-8) - 0 \cdot (-4))$$

$$= 3(0 - 40) + 6(18 - 20) + 7(-24)$$

$$= 3(-40) + 6(-2) + 7(-24)$$

$$= -120 - 12 - 168$$

$$= -300$$

$$\det(A_y) = 3 \cdot \begin{vmatrix} 3 & -5 \\ -4 & 6 \end{vmatrix} - 3 \cdot \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} + 7 \cdot \begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix}$$

$$= 3(3 \cdot 6 - (-5) \cdot (-4)) - 3(4 \cdot 6 - (-5) \cdot 5) + 7(4 \cdot (-4) - 3 \cdot 5)$$

$$= 3(18 - 20) - 3(24 + 25) + 7(-16 - 15)$$

$$= 3(-2) - 3(49) + 7(-31)$$

$$= -6 - 147 - 217$$

$$= -370$$

$$\text{det}(A_z) = 3 \cdot \begin{vmatrix} 0 & 3 \\ -8 & -4 \end{vmatrix} - (-6) \cdot \begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix}$$

$$= 3(0 \cdot (-4) - 3 \cdot (-8)) + 6(4 \cdot (-4) - 3 \cdot 5) + 3(4 \cdot (-8) - 0 \cdot 5)$$

$$= 3(0 + 24) + 6(-16 - 15) + 3(-32 - 0)$$

$$= 3(24) + 6(-31) + 3(-32)$$

$$= 72 - 186 - 96$$

$$= -210$$

Now, we can find the values of $x$, $y$, and $z$ using Cramer's rule:

$$x = \frac{\det(A_x)}{\det(A)} = \frac{-300}{-50} = 6$$

$$y = \frac{\det(A_y)}{\det(A)} = \frac{-370}{-50} = \frac{37}{5}$$

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-210}{-50} = \frac{21}{5}$$

Therefore, the solution to the system of linear equations is:

$$x = 6, \quad y = \frac{37}{5}, \quad z = \frac{21}{5}$$

□

**Homework of Cramer's Rule**

1. Solve the system of linear equations using Cramer's rule

$$\begin{cases} 3x + 5y = 2, \\ -x + 2y = 0. \end{cases}$$

2. Solve the system of linear equations using Cramer's rule

$$\begin{cases} 2x + y - 3z = 1, \\ 5x + 2z - 6z = 5, \\ 3x - y - 4z = 7. \end{cases}$$

## 8.4 Inverse Matrix Method

To solve a system of linear equations using the inverse matrix method, consider the system:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1, \\ a_{21}x + a_{22}y + a_{23}z = b_2, \\ a_{31}x + a_{32}y + a_{33}z = b_3. \end{cases}$$

This system can be represented in matrix form as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

To find **x** using the inverse of **A**, we need to have $\det(\mathbf{A}) \neq 0$. When the inverse of **A**, denoted $\mathbf{A}^{-1}$, exists, the solution can be found by:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

**Steps for Solving Using the Inverse Matrix**

1. Compute the inverse of **A**:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\text{adj}(\mathbf{A}),$$

where $\text{adj}(\mathbf{A})$ is the adjugate (transpose of the cofactor matrix) of **A**.

2. Multiply $\mathbf{A}^{-1}$ by **b** to find **x**:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

**Example 8.6.** Solve the system of linear equations using Inverse Matrix Method

$$\begin{cases} x - 2y = 3, \\ 4x + 5y = 12. \end{cases}$$

*Sol.* To solve the following system of linear equations, we express it in matrix form $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}.$$

**Step 1: Compute the Inverse of Matrix A**

The inverse of a 2x2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

such that $ad - bc \neq 0$.

**Step 1.1: Calculate the Determinant of A**

For our matrix **A**:

$$\det(\mathbf{A}) = 1 \cdot 5 - (-2) \cdot 4 = 5 + 8 = 13.$$

Since the determinant is not zero, the inverse exists.

**Step 1.2: Calculate the Inverse**

Using the formula for the inverse of a 2x2 matrix:

$$\mathbf{A}^{-1} = \frac{1}{13} \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{2}{13} \\ -\frac{4}{13} & \frac{1}{13} \end{pmatrix}.$$

**Step 2: Solve for x**

Now we can find **x** using:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Calculating this gives:

$$\mathbf{x} = \begin{pmatrix} \frac{5}{13} & \frac{2}{13} \\ -\frac{4}{13} & \frac{1}{13} \end{pmatrix} \begin{pmatrix} 3 \\ 12 \end{pmatrix}.$$

Performing the matrix multiplication:

1. First row:

$$\frac{5}{13}(3) + \frac{2}{13}(12) = \frac{15}{13} + \frac{24}{13} = \frac{39}{13} = 3.$$

2. Second row:

$$-\frac{4}{13}(3) + \frac{1}{13}(12) = -\frac{12}{13} + \frac{12}{13} = 0.$$

Thus, we have:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

## Final Solution

Therefore, the solution to the system of equations is:

$$x = 3, \quad y = 0.$$

□

**Example 8.7.** Solve the system of linear equations using Inverse Matrix Method

$$\begin{cases} 3x - 6y + 7z = 3, \\ 4x - 5z = 3, \\ 5x - 8y + 6z = -4. \end{cases}$$

*Sol.* We can represent this system in matrix form as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \begin{pmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}.$$

The determinant of $A$ is:

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} - (-6) \cdot \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} + 7 \cdot \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix}$$

$$= 3(0 \cdot 6 - (-5) \cdot (-8)) + 6(4 \cdot 6 - (-5) \cdot 5) + 7(4 \cdot (-8) - 0 \cdot 5)$$

$$= 3(0 - 40) + 6(24 + 25) + 7(-32 - 0)$$

$$= 3(-40) + 6(49) + 7(-32)$$

$$= -120 + 294 - 224$$

$$= -50$$

To find the adjugate (adjoint) of matrix **A**, we need to find the cofactor matrix and then transpose it.

1.Calculate the minor matrices for each element:

- For $a_{11} = 3$:

$$M_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} = 0 \cdot 6 - (-5)(-8) = -40$$

- For $a_{12} = -6$:

$$M_{12} = (-1)^{1+2} \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} = -(4 \cdot 6 - (-5) \cdot 5) = -49$$

- For $a_{13} = 7$:

$$M_{13}(-1)^{1+3} = \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix} = 4(-8) - 0 \cdot 5 = -32$$

- For $a_{21} = 4$:

$$M_{21} = (-1)^{2+1} \begin{vmatrix} -6 & 7 \\ -8 & 6 \end{vmatrix} = -((-6)(6) - 7(-8)) = -20$$

- For $a_{22} = 0$:

$$M_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 7 \\ 5 & 6 \end{vmatrix} = 3 \cdot 6 - 7 \cdot 5 = -17$$

- For $a_{23} = -5$:

$$M_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -6 \\ 5 & -8 \end{vmatrix} = -(3(-8) - (-6)5) = -6$$

- For $a_{31} = 5$:

$$M_{31} = (-1)^{3+1} \begin{vmatrix} -6 & 7 \\ 0 & -5 \end{vmatrix} = (-6)(-5) - 7 \cdot 0 = 30$$

- For $a_{32} = -8$:

$$M_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 7 \\ 4 & -5 \end{vmatrix} = -(3(-5) - 7 \cdot 4) = 43$$

- For $a_{33} = 6$:

$$M_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = 3 \cdot 0 - (-6)(4) = 24$$

2. Form the cofactor matrix:

$$\mathbf{C} = \begin{bmatrix} -40 & -49 & -32 \\ -20 & -17 & -6 \\ 30 & 43 & 24 \end{bmatrix}$$

3. Transpose the cofactor matrix to obtain the adjugate (adjoint) matrix**:

$$\text{adj}(\mathbf{A}) = \mathbf{C}^\top = \begin{bmatrix} -40 & -20 & 30 \\ -49 & -17 & 43 \\ -32 & -6 & 24 \end{bmatrix}$$

5. Find the inverse by dividing the adjugate by the determinant:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{-50} \begin{bmatrix} -40 & -20 & 30 \\ -49 & -17 & 43 \\ -32 & -6 & 24 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{-40}{-50} & \frac{-20}{-50} & \frac{30}{-50} \\ \\ \frac{-49}{-50} & \frac{-17}{-50} & \frac{43}{-50} \\ \\ \frac{-32}{-50} & \frac{-6}{-50} & \frac{24}{-50} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} & -\frac{3}{5} \\ \\ \frac{49}{50} & \frac{17}{50} & -\frac{43}{50} \\ \\ \frac{16}{25} & \frac{3}{25} & -\frac{12}{25} \end{bmatrix}$$

Now we can find **x** using:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

$$\mathbf{x} = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} & -\frac{3}{5} \\ \\ \frac{49}{50} & \frac{17}{50} & -\frac{43}{50} \\ \\ \frac{16}{25} & \frac{3}{25} & -\frac{12}{25} \end{pmatrix} \begin{pmatrix} 3 \\ \\ 3 \\ \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{4}{5}(3) + \frac{2}{5}(3) - \frac{3}{5}(-4) \\ \frac{49}{50}(3) + \frac{17}{50}(3) - \frac{43}{50}(-4) \\ \frac{16}{25}(3) + \frac{3}{25}(3) - \frac{12}{25}(-4) \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} \frac{4}{5}(3) + \frac{2}{5}(3) - \frac{3}{5}(-4) \\ \frac{49}{50}(3) + \frac{17}{50}(3) - \frac{43}{50}(-4) \\ \frac{16}{25}(3) + \frac{3}{25}(3) - \frac{12}{25}(-4) \end{pmatrix} = \begin{pmatrix} 6 \\ \frac{37}{5} \\ \frac{21}{5} \end{pmatrix}$$

Therefore, the solution to the system of linear equations is:

$$x = 6, \quad y = \frac{37}{5}, \quad z = \frac{21}{5}$$

□

**Homework of Inverse Matrix Method**

1. Solve the system of linear equations using Inverse Matrix Method

$$\begin{cases} x - 2y = 3, \\ 4x + 5y = 12. \end{cases}$$

2. Solve the system of linear equations using Inverse Matrix Method

$$\begin{cases} 2x + y - 3z = 1, \\ 5x + 2z - 6z = 5, \\ 3x - y - 4z = 7. \end{cases}$$

# Lecture 11

## 9 Least Squares Approximations

Given a set of data points $(x_i, y_i)$ for $i = 1, 2, \ldots, n$, the least squares approximation seeks to find the line $y = mx + b$ that minimizes the sum of the squares of the vertical distances from the points to the line.

The line of best fit is determined by solving the following system of equations for $m$ and $b$:

$$\sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i$$
$$\sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + nb$$

Solving these equations, we get:

$$m = \frac{n \sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$
$$b = \frac{\sum_{i=1}^{n} y_i - m \sum_{i=1}^{n} x_i}{n}$$

Thus, the equation of the line of best fit is:

$$y = mx + b$$

**Example 9.1.** Find the following points to linear form $y = a + bx$, where

$$(x_1, y_1) = (1, 3)$$

$$(x_2, y_2) = (2, 5)$$

$$(x_3, y_3) = (3, 8)$$

$$(x_4, y_4) = (4, 13)$$

$$(x_5, y_5) = (5, 16)$$

*Sol.* To find the linear equation in the form $y = a + bx$, we will calculate the slope $b$ and the intercept $a$.

**Step 1: Calculate the Necessary Sums**

$$n = 5$$

$$\sum x_i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum y_i = 3 + 5 + 8 + 13 + 16 = 45$$

$$\sum x_i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$\sum x_i y_i = 1 \cdot 3 + 2 \cdot 5 + 3 \cdot 8 + 4 \cdot 13 + 5 \cdot 16 = 3 + 10 + 24 + 52 + 80 = 169$$

**Step 2: Calculate $b$ and $a$**

The formulas for the slope $b$ and intercept $a$ are given by:

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i - b \sum x_i}{n}$$

Calculating $b$:

$$b = \frac{5 \cdot 169 - 15 \cdot 45}{5 \cdot 55 - 15^2}$$

Simplifying:

$$b = \frac{845 - 675}{275 - 225} = \frac{170}{50} = 3.4$$

Calculating $a$:

$$a = \frac{45 - 3.4 \cdot 15}{5} = \frac{45 - 51}{5} = \frac{-6}{5} = -1.2$$

**Final Linear Equation**

Thus, the linear equation that best fits the given data points is:

$$y = -1.2 + 3.4x$$

$\square$

**Example 9.2.** Find the following points to linear form $y = ae^{bx}$, where

$$(x_1, y_1) = (0, 1.5)$$

$$(x_2, y_2) = (1, 2.5)$$

$$(x_3, y_3) = (2, 3.5)$$

$$(x_4, y_4) = (3, 5)$$

$$(x_5, y_5) = (4, 7.5)$$

*Sol.* To find the exponential equation in the form $y = ae^{bx}$, we will linearize the equation using the natural logarithm.

**Step 1: Linearization**

Taking the natural logarithm of both sides:

$$\ln(y) = \ln(a) + bx$$

Setting $Y = \ln(y)$, $A = \ln(a)$, $B = b$, and $X = x$.

**Step 2: Calculate the Data Points**

We are given the following data points:

$$x_1 = 0, \qquad y_1 = 1.5 \qquad \Rightarrow Y_1 = \ln(1.5) \approx 0.4055$$

$$x_2 = 1, \qquad y_2 = 2.5 \qquad \Rightarrow Y_2 = \ln(2.5) \approx 0.9163$$

$$x_3 = 2, \qquad y_3 = 3.5 \qquad \Rightarrow Y_3 = \ln(3.5) \approx 1.2528$$

$$x_4 = 3, \qquad y_4 = 5 \qquad \Rightarrow Y_4 = \ln(5) \approx 1.6094$$

$$x_5 = 4, \qquad y_5 = 7.5 \qquad \Rightarrow Y_5 = \ln(7.5) \approx 2.0149$$

**Step 3: Calculate $b$ and $a$**

The necessary sums are:

$$n = 5$$

$$\sum x_i = 0 + 1 + 2 + 3 + 4 = 10$$

$$\sum y_i \approx 0.4055 + 0.9163 + 1.2528 + 1.6094 + 2.0149 \approx 6.1989$$

$$\sum x_i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum x_i y_i \approx 0 + 0.9163 + 2.5056 + 4.8282 + 8.0596 \approx 16.3097$$

Using the formulas:

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

Calculating $b$ and $A$:

$$b = \frac{5 \cdot 16.3097 - 10 \cdot 6.1989}{5 \cdot 30 - 10^2}$$

Simplifying:

$$b = \frac{81.5485 - 61.989}{150 - 100} = \frac{19.5595}{50} \approx 0.3912$$

$$A = \frac{\sum Y_i - b \sum x_i}{n}$$

$$A = \frac{6.1989 - 0.3912 \cdot 10}{5} = \frac{6.1989 - 3.912}{5} = \frac{2.2869}{5} \approx 0.4574$$

Calculating $a$:

$$a = e^A = e^{0.4574} \approx 1.59$$

**Final Linear Equation**

Thus, the exponential equation that best fits the given data points is:

$$y = ae^{bx}$$

$$y = 1.59e^{0.3912x}$$

□

**Homework of Least Squares Approximations**

1. Find the following points to linear form $y = a + bx$, where

$$(x_1, y_1) = (1, 1)$$
$$(x_2, y_2) = (2, 5)$$
$$(x_3, y_3) = (3, 7)$$
$$(x_4, y_4) = (4, 9)$$
$$(x_5, y_5) = (5, 11)$$

2. Find the following points to linear form $y = ae^{bx}$, where

$$(x_1, y_1) = (0, 0.5)$$
$$(x_2, y_2) = (1, 1)$$
$$(x_3, y_3) = (2, 1.5)$$
$$(x_4, y_4) = (3, 2)$$
$$(x_5, y_5) = (4, 2.5)$$

# Lecture 12

## 10 Introduction to Fourier Series

In mathematics, the Fourier series is a method for expressing a periodic function as a sum of sine and cosine functions. The concept was introduced by Jean-Baptiste Joseph Fourier in the early 19th century, and it has since become a fundamental tool in various fields such as signal processing, physics, and engineering.

**Definition 10.1 (Periodic Functions).** A function $f(x)$ is called periodic if there exists a positive number $T$ such that:

$$f(x+T) = f(x)$$

for all values of $x$. The smallest positive value of $T$ is called the fundamental period of the function.

**For example:**

1. **Sine and Cosine Functions:**

$$\sin(x+2\pi) = \sin(x), \quad \cos(x+2\pi) = \cos(x)$$

The period of both the sine and cosine functions is $2\pi$.

2. **Square Wave:**

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < \pi \\ -1, & \text{if } \pi \leq x < 2\pi \end{cases}$$

This function has a period of $2\pi$.

**Definition 10.2 (Fourier series).** Given a periodic function $f(x)$ with period $2\pi$, the Fourier series of $f(x)$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the coefficients $a_0$, $a_n$, and $b_n$ are defined as:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

*Remark* 10.1. If the function $f(x)$ defined on interval $-\pi < x < \pi$, then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

**Example 10.1.** Find Fourier series of the function $f(x) = x$, from$x = 0$ to $x = 2\pi$

*Sol.* To find the Fourier series of $f(x) = x$ defined on the interval $[0, 2\pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

**Step 1: Calculating $a_0$**

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x\, dx = \frac{1}{2\pi} \left. \frac{x^2}{2} \right|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{(2\pi)^2}{2} = \pi$$

**Step 2: Calculate** $a_n$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)\,dx$$

Using integration ($uv - \int vdu$) by parts, let $u = x$ and $dv = \cos(nx)\,dx$:

$$du = dx, \quad v = \frac{\sin(nx)}{n}$$

$$a_n = \frac{1}{\pi}\left[\frac{x\sin(nx)}{n}\bigg|_0^{2\pi} - \int_0^{2\pi}\frac{\sin(nx)}{n}\,dx\right] = 0$$

since the integral of $\sin(nx)$ over $[0, 2\pi]$ is zero

**Step 3: Calculate** $b_n$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

Using integration by parts, let $u = x$ and $dv = \sin(nx)\,dx$:

$$du = dx, \quad v = -\frac{\cos(nx)}{n}$$

$$b_n = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\bigg|_0^{2\pi} + \int_0^{2\pi}\frac{\cos(nx)}{n}\,dx\right] = \frac{1}{\pi}\left[\frac{2\pi}{n}\right] = -\frac{2}{n}$$

So, the Fourier series for $f(x) = x$ in the interval $[0, 2\pi]$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$f(x) = \pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin(nx)$$

□

**Homework 1: Fourier Series**

Find Fourier series of the function $f(x) = 5x$, from $x = 0$ to $x = 2\pi$

# Lecture 13

## 10.1 Fourier series of Even and Odd Function

Fourier series can be used to represent periodic functions as the sum of sines and cosines. The form of the series depends on whether the function is even, odd, or neither.

**Definition 10.3 (Even Function).** A function $f(x)$ is called an even function if it satisfies the following condition for all $x$ in its domain:

$$f(-x) = f(x).$$

For example, with $f(x) = x^2$:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Thus, $f(x) = x^2$ is an even function.

**Definition 10.4 (Odd Function).** A function $f(x)$ is called an odd function if it satisfies the following condition for all $x$ in its domain:

$$f(-x) = -f(x).$$

Some common examples of odd functions are:

- $f(x) = x^3$
- $f(x) = \sin(x)$
- $f(x) = x$

For example, with $f(x) = x^3$:

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Thus, $f(x) = x^3$ is an odd function.

### 10.1.1 Fourier Series for Even Function

Given a function $f(x)$, the Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where } b_n = 0$$

**Example 10.2.** Find Fourier series of the function $f(x) = x^2$, from$x = 0$ to $x = 2\pi$

*Sol.* To find the Fourier series of $f(x) = x$ defined on the interval $[0, 2\pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

**Since** $f(x) = x^2$ **is even** then $b_n = 0$.

**Calculate** $a_0, a_n$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{(2\pi)^3}{3} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx)\,dx$$

Using integration by parts (let $u = x^2$ and $dv = \cos(nx)dx$):

$$u = x^2 \Rightarrow du = 2xdx, \qquad dv = \cos(nx)dx \Rightarrow v = \frac{\sin(nx)}{n}$$

$$\int x^2 \cos(nx)dx = x^2 \cdot \frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n} \cdot 2xdx$$

$$\int x^2 \cos(nx)dx = x^2 \cdot \frac{\sin(nx)}{n} - 2\int \frac{x\sin(nx)}{n}dx$$

Again using integration by parts on the remaining integral (let $u = x$ and $dv = \sin(nx)dx$):

$$u = x \quad \Rightarrow \quad du = dx$$

$$dv = \sin(nx)dx \quad \Rightarrow \quad v = -\frac{\cos(nx)}{n}$$

$$= x^2 \cdot \frac{\sin(nx)}{n} - 2\left(x \cdot -\frac{\cos(nx)}{n^2} - \int -\frac{\cos(nx)}{n^2}dx\right)$$

$$= x^2 \cdot \frac{\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2}{n^2}\int \cos(nx)dx$$

$$= x^2 \cdot \frac{\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2}{n^3}\sin(nx)$$

Evaluating this integral from $0$ to $2\pi$:

$$a_n = \frac{1}{\pi}\left[\left(\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right)\Bigg|_0^{2\pi}\right] = -\frac{4}{n^2}$$

The Fourier series of $f(x) = x^2$ is:

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}\cos(nx)$$

□

### 10.1.2 Fourier Series for Odd Function

Given a function $f(x)$, the Fourier series is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where } a_0, a_n = 0$$

**Example 10.3.** Find Fourier series of the function $f(x) = x$, from$x = -\pi$ to $x = \pi$

*Sol.* To find the Fourier series of $f(x) = x$ defined on the interval $[-\pi, \pi]$, we express $f(x)$ as a Fourier series in the form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

**Since** $f(x) = x$ **is odd** then $a_0 = a_n = 0$.

**Calculate** $b_n$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx$$

Using integration by parts:

$$\int x \sin(nx)\, dx = -\frac{x \cos(nx)}{n}\bigg|_{-\pi}^{\pi} + \int \frac{\cos(nx)}{n}\, dx$$

Evaluating this integral:

$$= -\frac{\pi \cos(n\pi)}{n} + \frac{(-\pi)\cos(-n\pi)}{n} + \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= -\frac{2\pi \cos(n\pi)}{n} = -\frac{2\pi(-1)^n}{n}$$

Thus,

$$b_n = \frac{1}{\pi}\left(-\frac{2\pi(-1)^n}{n}\right) = \frac{2(-1)^n}{n}$$

So, the Fourier series for $f(x) = x$ in the interval $[0, 2\pi]$ is:

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin(nx)) \quad \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin(nx)$$

□

## 10.2 Half-Range Series

The half-range Fourier series allows us to represent a function using either sines or cosines. This is particularly useful when dealing with functions that are defined only on a finite interval.

### 10.2.1 Half-Range Sine Series

For a function $f(x)$, the half-range sine series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (b_n \sin(nx))$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin(nx)\, dx$$

### 10.2.2 Half-Range Cosine Series

For a function $f(x)$ defined on $0 \leq x \leq \pi$, the half-range cosine series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where}$$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

**Homework 2: Fourier Series**

1. Find Fourier series of the function $f(x) = 4x^2$, from$x = 0$ to $x = 2\pi$

2. Find Fourier series of the function

$$f(x) = \begin{cases} 1 \text{ if } & 0 < x < \pi \\ 2 \text{ if } & \pi < x < 2\pi \end{cases}$$

3. Find cosine Half-range series for the function defined as

$$f(x) = x, \quad 0 < x < \pi$$

4. Find sine Half-range series for the function defined as

$$f(x) = x, \quad 0 < x < \pi$$