Lecture#3 Important Probability Distributions

1- Uniform Distribution

In uniform random variable the pdf f(x) is constant for the whole range of the continues random variable x. In general, the pdf is given by:

$$f(x) = \begin{cases} \dfrac{1}{(b-a)} & a < x < b \\ 0 & elsewhere \end{cases}$$

and: $\bar{x} = \frac{a+b}{2}$ $\quad \overline{x^2} = \frac{b^2+ab+a^2}{12}$ $\quad \sigma^2 = \overline{x^2} - \bar{x}^2$ (Verify)

An example of uniform source is the random binary source that has the probability of binary “0” identical to the probability of binary “1”.

Q- For a=2 and b=4, sketch f(x), then find $\bar{x}$, $\overline{x^2}$, and σ^2

2- Binomial Distribution (or Repeated Bernoulli Trials)

This is an example of discrete random variable. Any experiment with two outcomes only and repeated n times can be considered as Binomial distribution. This is the same expression used in binomial expansion and its combination coefficients in Lecture#3.

Let the two outcomes called the “Success event” with probability of P_s and the “Failure event” with probability of P_F. Clearly:
P_s = The probability of the success event in single trial and (P_s+ P_F) =1.

Define the random variable of the binomial distribution as (k), where:
k= Number of times that the success event occurred out of n trials

The probability that exactly the success event occurred for k times is given by:

$$P(k) = \binom{n}{k}.P_S^k P_F^{n-k} \quad \text{or} \quad P(k) = \frac{n!}{k!.(n-k)!} P_S^k (1-P_S)^{n-k}$$

In above equation k=0,1,2,3,…….n and $\sum_{all\ k} P(k) = 1$
With mean value: $\bar{k} = n.P_S$ and variance $\sigma^2 = n.P_S P_F$ also: $\overline{k^2} = \sigma^2 + (\bar{k})^2$

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Example

Consider an experiment of tossing fair die for 8 times, then find the probability of:

a) Exactly 4 times the outcome 3 occurred.
b) The outcome 5 occurred at least 6 times
c) An odd number occurred at most 7 times
d) No odd occurred

Solution:

Since we have repeated trials and the required probabilities are for the occurrence of certain outcome, then the experiment is Bernoulli trials. Thus, in each case we need to define the success outcome with $k\ and\ P_S$. Bearing in mind that the total number of trials in each case is n=8.

a) The success outcome or event is 3, while k=4 with P_S=P(outcome=3) in single trial. Thus, $P_S = 1/6$ and clearly $P_F = 5/6$

$$\therefore\ P(k=4) = \binom{n}{k}.P_S^k P_F^{n-k} = \binom{8}{4}.(\frac{1}{6})^4.(\frac{5}{6})^4 =$$

b) The success outcome or event is 5, while k=6 or 7 or 8 with P_S=P(outcome=5) in single trial. Thus, $P_S = 1/6$ and clearly $P_F = 5/6$

$$\therefore\ P(k = 6\ or\ 7\ 0r\ 8) = P(k=6) + P(k=7) + P(k=\ 8)$$

$$= \sum_{k=6,7,8} \binom{n}{k}.P_S^k P_F^{n-k}$$

$$= \binom{8}{6}.(\frac{1}{6})^6.(\frac{5}{6})^2 + \binom{8}{7}.(\frac{1}{6})^7.(\frac{5}{6})^1 + \binom{8}{8}.(\frac{1}{6})^8.(\frac{5}{6})^0$$

c) The success outcome or event is "Odd Number", while k=0,1,2…7, with P_S=P(outcome=Odd) in single trial. Thus, $P_S = \frac{3}{6} = 1/2$ and clearly $P_F = 1/2$.

$$\therefore\ P(k = 0,1,2\ ....7) = P(k=0) + \cdots..+P(k=7) = \sum_{k=0,1,2...7} \binom{n}{k}.P_S^k P_F^{n-k}$$

The above is very complex to calculate, then one may use the following instead:

$$P(k = 0,1,2\ ....7) = 1 -\ P(k=8) = 1 - \binom{8}{8}.(\frac{1}{2})^8.(\frac{1}{2})^0 = 1 - 2^{-8}\quad \text{(check)}$$

d) No odd occurred (you solve it)

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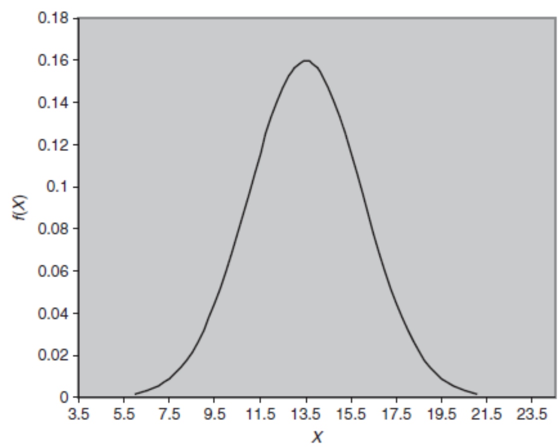
3- Normal or Gaussian Distribution

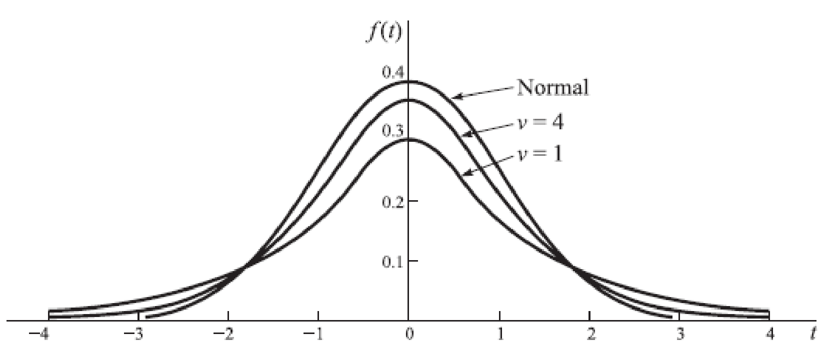
The normal or Gaussian pdf is the most known distribution since it models almost all-natural behavior of experiment or sample space available in normal life. The expression of f(x) takes the Gaussian shape (sometimes called bell shape) and the expression is governed by the variance and the mean value of x. Gaussian or Normal distribution pdf is given by:

$$f(x)= \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\overline{x})^2/2\sigma^2} \quad for \quad -\infty < x < \infty$$

The following is a plot of normal pdf f(x) with mean $\overline{x} = 13.5$ and standard deviation $\sigma = 2.5$ as an example:



The change in the mean affect the positioning of the mean (shift of the curve to the left or right). The peak value in f(x)-axis of the distribution is determined by $\frac{1}{\sigma\sqrt{2\pi}}$ while the center of the peak in the x-axis is at the mean $\overline{x}$ $(i.e\ at\ x = \overline{x})$. The following Normal curves with different σ and the mean is fixed at $\overline{x} = 0$.



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In any case (of the mean or the variance) the total area under normal curve is unity since it is a pdf, so:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\overline{x})^2/2\sigma^2}\, dx \;= 1$$

4- The Poisson Distribution

A discrete random variable taking unbounded positive values obeys a Poisson law with parameter A if its distribution is given by:

$$P(k) = \frac{A^k}{k!}\, e^{-A} \quad \text{for k=0,1,2,3……}$$

With mean and variance given $\overline{x} = A$ *and* $\sigma^2 = A$

The main application of Poisson distribution is related to queuing applications as in network access trials and cell phone calls (k is the number of users asking for the connection service. It is also an accurate approximation of binomial distribution for the case where P_S *or* P_F become very small and n very large.

5- The Exponential Distribution

Exponential distribution has a quite distinct position in traffic, queueing and reliability domains. The distribution depends on a single parameter, traditionally denoted as μ in the tele traffic field, and the density function is given by:

$$f(x) = \mu e^{-\mu x}, \quad \text{for } x \geq 0.$$

Mean value: $m = \dfrac{1}{\mu}$,

Variance: $\text{Var} = \dfrac{1}{\mu^2}$.

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