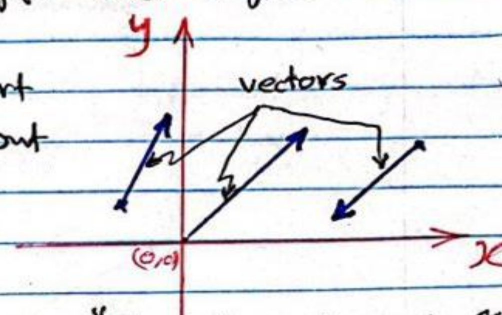
Vectors :- المتجهات

- The vector is only two pieces of information:
1- Direction
2- Length or Magnitude

المتجه عبارة عن مكونين وهما : اتجاه وطول أو كمية .

- We can graph a vector by an arrow that we can visualize on x-y plane and we can capture it by the arrow length and angle

- Vectors on graph could start From not just an origion, but From any where -



"Examples of Vectors"

Examples of Vectors :-

- To answer the question "What is the current temperature?" We use a single number (scalar); likewise the question about a mass ;
- While to answer the question "What is the current velocity of the wind?", we need more than just a single number. We need magnitude (speed) and direction. This where vectors come to handy.

Position, displacement, velocity, acceleration, Force, momentum & torque are all physical quantities that can be represented mathmatically by vectors.

Vector Denoting :-

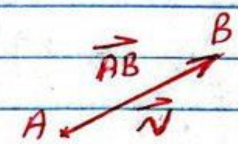
- Vectors are writing with arrow on top on equations.

Ex)

- Velocity vector $\longrightarrow \vec{v}$
- Force vector $\longrightarrow \vec{F}$

Note) Any variable symbol with no arrow on top means scalar.

- A vector can be geometrically represented by a direction line segment with a head & a tail ;



so vector $\vec{AB}$ is a vector from point A to B.

- Also, we can denote vector $\vec{AB}$ by a small case letter $\vec{v}$
- The length of the arrow A B Corresponds to the magnitude of the vector.
- The arrow points in the direction of the vector.

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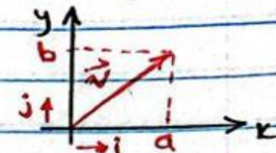
How to represent the vector mathmatically :-

Vector in plane :- كيف تمثل المتجه رياضيا

We can write vectors as Columns. Let us take a very important special vector as example :-

$i = \begin{pmatrix}1\\0\end{pmatrix}$, $j = \begin{pmatrix}0\\1\end{pmatrix}$

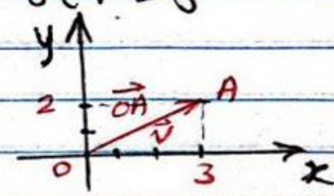


Any vector in the xy-plane can be written in terms of i & j using the triangle law & scalar multiplication.

$$\vec{v} = ai + bj = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

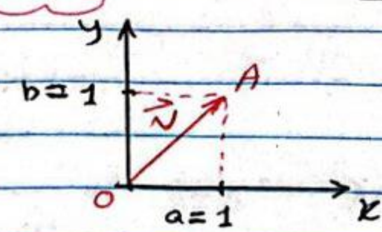
Ex)

$$\begin{pmatrix}3\\2\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix} = 3i + 2j$$



Notes

* IF $a = b = 1$, then $\vec{v} = i + j$ is a "unit vector"



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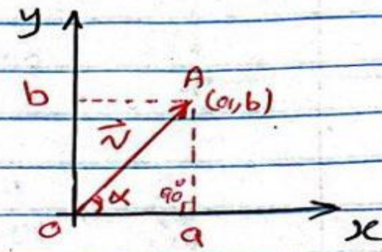
Finding the length/magnitude and the direction of vector :- إيجاد طول (مقدار) و اتجاه المتجه

If $\vec{v} = a\,i + b\,j$ ---- ①, then the length/magnitude of a vector $\vec{v}$ is :-



$$|v| = \sqrt{a^2 + b^2} \quad \text{---- ②}$$

(نظرية فيثاغورس)

- It's a Pythagorean theorem

$$\left.\begin{aligned} a &= |v| \cos\alpha \\ b &= |v| \sin\alpha \end{aligned}\right\} \text{---- ③}$$

$$\tan\alpha = \frac{b}{a}$$

Substitute eq. ③ in ① yields ;

$$\vec{v} = |v| \left(\cos\alpha \; i + \sin\alpha \; j \right)$$

$\vec{v}$ — vector symbol
$|v|$ — vector length
i, j — unit vector components (basis / Fundamental vector components)
α — vector angle with x-axis

Ex/ Find a vector in plane of length (7 units) & makes angle (35°) with x-axis ?

Solution

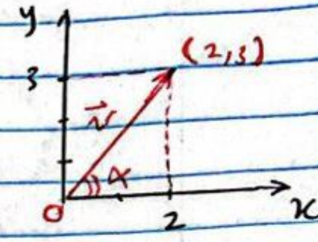
since $|\vec{v}| = 7$ & $\alpha = 35°$

$$\therefore \vec{v} = 7 \times (\cos 35 \; i + \sin 35 \; j)$$

$$\boxed{\vec{v} = 5.7\,i + 4\,j} \quad \text{Ans}$$

Ex) Find the angle between the vector $\vec{v} = 2i + 3j$ and the x-axis?

Solution



$|\vec{v}| = \sqrt{a^2+b^2}$; $a = 2$, $b = 3$

$|\vec{v}| = \sqrt{2^2+3^2} = \sqrt{13}$

$a = |v| \cos\alpha \Longrightarrow \cos\alpha = \dfrac{a}{|v|} = \dfrac{2}{\sqrt{13}}$

$\therefore \alpha = \cos^{-1} \dfrac{2}{\sqrt{13}} = \boxed{56.3^\circ}$ Ans

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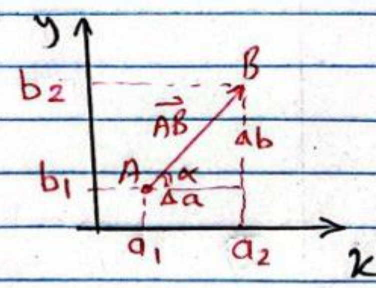
Vectors with tail not in origin :-

متجهات نقطة ابتداءه ليس ع (0,0)

Vectors can be start not from the origion, but from any where. like A to B



$\therefore \vec{AB} = \Delta a\, i + \Delta b\, j$

$\vec{AB} = (a_2 - a_1)\, i + (b_2 - b_1)\, j$

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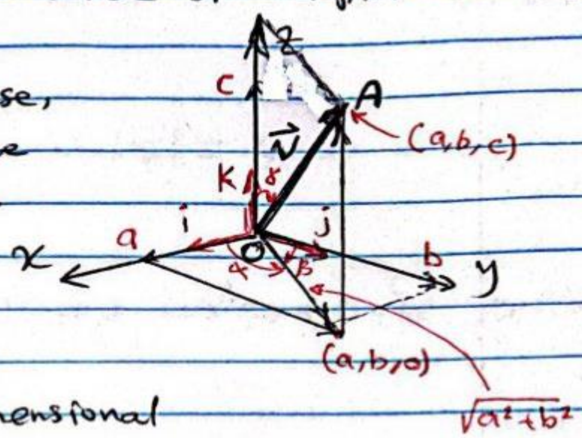
Vectors in a space :-

- This could be in three or higher dimensions

- Similar to the 2-D case, but we now have three basis vectors i, j & k.
- From these three components unit vectors we can describe any vector in three-dimensional space.



$$\vec{V} = \vec{OA} = a\,i + b\,j + c\,K,$$

where :-

i, j, K = Basis or Fundamental unit vector.
a, b, c = Directional numbers (scalars).
α, β, γ = Directional angles.

$$|V| = |OA| = \sqrt{a^2+b^2+c^2},$$

$$a = |V| \times \cos\alpha$$
$$b = |V| \times \cos\beta$$
$$c = |V| \times \cos\gamma$$

$$\frac{\vec{V}}{|V|} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$ = unit vector with direction of $\vec{V}$

And,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

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Ex) Find a vector in space of length (5 units) that makes angles (70°) with x-axis, (85°) with y-axis?

Solution

$\alpha = 70^\circ \quad , \beta = 85^\circ \quad , |V| = 5$

$\gamma = ? \quad , \vec{V} = ?$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \cos^2 70 + \cos^2 85 + \cos^2\gamma = 1$

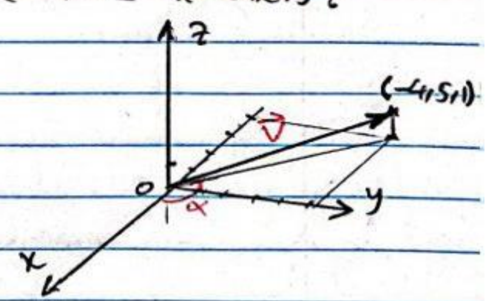
$\therefore \boxed{\cos\gamma = 0.935}$

$\therefore \vec{V} = |V|(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$

$= 5(\cos 70\, i + \cos 85\, j + 0.935\, k)$

$\boxed{\vec{V} = 1.7\, i + 0.436\, j + 4.675\, k}$ Ans

Ex) Find the angle between the vector $\vec{V} = -4i + 5j + k$ and the x-axis?



Solution

$a = -4 \quad , b = 5 \quad , c = 1$

$|V| = \sqrt{a^2 + b^2 + c^2}$

$|V| = \sqrt{(-4)^2 + (5)^2 + (1)^2} = \sqrt{42}$

$\cos\alpha = \frac{a}{|V|} \Rightarrow \alpha = \cos^{-1}\frac{a}{|V|} = \cos^{-1}\frac{-4}{\sqrt{42}}$

$\boxed{\alpha = 128^\circ}$

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Scalar product :- (Dot product)

ضرب القيم العددية

Let $\vec{A} = a_1 i + a_2 j + a_3 k$

And $\vec{B} = b_1 i + b_2 j + b_3 k$

Then, $\vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta$

where θ — is the angle between $\vec{A}$ & $\vec{B}$

Properties :-

1- $\vec{A} \cdot \vec{A} = |A|^2$

2- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3- $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A|\,|B|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} + \sqrt{b_1^2 + b_2^2 + b_3^2}}$

4- $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$ [Orthogonal Vectors]

متجهات متعامدة

5- $ai + bj \perp bi - aj$

Ex) Find the angle "θ" between $\vec{A} = i - 2j - 2k$ & $\vec{B} = 6i + 3j + 2k$?

Solution

$\vec{A} \cdot \vec{B} = (1 * 6) + (-2 * 3) + (-2 * 2) = \boxed{-4}$

$|A| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = \boxed{3}$

$|B| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = \boxed{7}$

$\Rightarrow |A|\,|B| = \boxed{21}$

$\therefore \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A|\,|B|} \Rightarrow \theta = \cos^{-1} \dfrac{-4}{21} \approx \boxed{101^\circ}$ **Ans**

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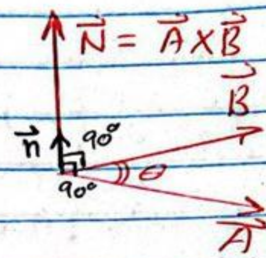
Vector Product :- (Cross-product)

ضرب المتجهات

Normal vector is what yields from vector product or cross product.

$$\boxed{\vec{N} = \vec{A} \times \vec{B} = \vec{n}\,|A|\,|B|\sin\theta}$$

where $\vec{n}$ — is a normal unit vector



$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad ; \text{where, } \begin{array}{l} \vec{A} = a_1 i + a_2 j + a_3 k \\ \vec{B} = b_1 i + b_2 j + b_3 k \end{array}$$

Properties :-

1- $\vec{A} \times \vec{A} = 0 \longrightarrow$ "$\sin 0 = 0$"

2- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

3- $\vec{A} \parallel \vec{B} \Longrightarrow \vec{A} \times \vec{B} = 0 \longrightarrow \sin 0 = 0$

4- Area of $\Delta\, ABC = \frac{1}{2}|\vec{A} \times \vec{B}|$

Ex) Find $\vec{A}\times\vec{B}$ & $\vec{B}\times\vec{A}$ if

$\vec{A} = 2i + j + k$

$\vec{B} = -4i + 3j + k$

Solution)

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= (1\times1 - (3\times1))\,i - (2\times1 - (-4\times1))\,j + (2\times3 - (-4\times1))\,k$$

$$\boxed{\vec{A}\times\vec{B} = -2i - 6j + 10k}$$

but,

$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B} = \boxed{2i + 6j - 10k}$$

Triple Product :- الضرب الثلاثي المكرر

A- Scalar triple product :-

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

Note)

1- Box Volume is $\Rightarrow$ $V_{box} = |\vec{A}\cdot\vec{B}\times\vec{C}|$

2- Pyramid Volume is $\longrightarrow$ $V_{Py} = \frac{1}{6}|\vec{A}\cdot\vec{B}\times\vec{C}|$

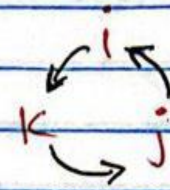
B- Vector triple Product :-

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\times B - (\vec{A}\cdot\vec{B})\,\vec{C}$$

Note)

$i\cdot i = j\cdot j = k\cdot k = 1$; $i\times i = j\times j = k\times k = 0$

$i \cdot j = j \cdot k = k \cdot i = 0$



$i \times j = k$
$j \times k = i$
$k \times i = j$

HW#2

1- Find the Length & direction of these vectors & the angles make with the x-axis?

a- $5i + 12j$
b- $\sqrt{3}\,i + j$

2- Find a vector 6 units long in the direction of $\vec{A} = 2i + 2j - k$

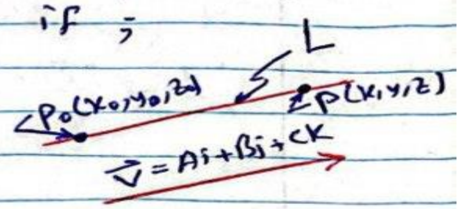
3- Find the area of the triangle whose vertices are $A(1,-1,0)$, $B(2,1,-1)$, $C(-1,1,2)$?

4- If $\vec{A} = 2i - j$ & $B = i + 3j - 2k$, find $A \times B$, then calculate $(A \times B) \cdot A$?

Equation of line in a space — معادلة الخط المستقيم في الفضاء

If L is a line in a space that passes through a point $P_0(x_0, y_0, z_0)$ & it parallel to a vector $\vec{V} = Ai + Bj + Ck$, Then $P(x, y, z)$ is any point lies on L only if ;

$$\overrightarrow{P_0P} = t\vec{V} \quad \text{--- (1)}$$



Where ;

t — time parameter

So eq. (1) can be written as ;

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(Ai + Bj + Ck)$$

$x - x_0 = At \longrightarrow x = At + x_0$

$y - y_0 = Bt \longrightarrow y = Bt + y_0$

$z - z_0 = Ct \longrightarrow z = Ct + z_0$ ← Eq. of line in a space

Ex

Find parametric equations for the line through the point $(-2, 0, 4)$ parallel to the vector $\vec{V} = 2i + 4j - 2k$

Solution

$P_0(x_0, y_0, z_0) = (-2, 0, 4)$

$Ai + Bj + Ck = 2i + 4j - 2k$

$\therefore \boxed{x = 2t - 2}$

$\boxed{y = 4t}$; $\boxed{z = -2t + 4}$

Ex)

Find parametric equations for line through the points $P(-3,2,-3)$ & $Q(1,-1,4)$?

(solution)

$$\overrightarrow{PQ} = (1-(-3))\,i + (-1-2)\,j + (4-(-3))\,k$$

$$\boxed{\overrightarrow{PQ} = 4i - 3j + 7k} = \boxed{Ai + Bj + Ck}$$

* let $(x_0, y_0, z_0) = (-3, 2, -3)$

$\therefore \boxed{x = 4t - 3}$; $\boxed{y = -3t + 2}$; $\boxed{z = 7t - 3}$

or;

* let $(x_0, y_0, z_0) = (1, -1, 4)$

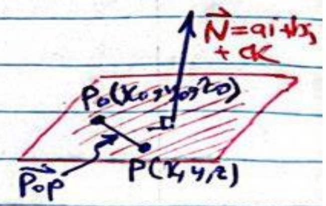
$\therefore \boxed{x = 4t + 1}$; $\boxed{y = -3t + 1}$; $\boxed{z = 7t + 4}$

Equation of the plane : معادلة المستوي

To find the eq. of the plane that passes through the point $P_0(x_0, y_0, z_0)$ & it's normal vector is $\vec{N} = ai + bj + ck$

Let $p(x, y, z)$ be any point in the plane

$$\overrightarrow{P_0P} = (x - x_0)\,i + (y - y_0)\,j + (z - z_0)\,k$$



$$P_0P \perp N \Longrightarrow \overrightarrow{P_0P} \cdot \vec{N} = 0$$

$$\Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Longrightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

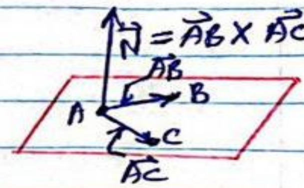
or

$$ax + by + cz = d \longleftarrow \text{Equation of the plane}$$

Ex)

Find the equation of the plane having the points $A(2,3,5)$, $B(7,2,1)$ & $C(1,1,1)$

Solution



في هذا المثال توجد نقاط ثلاث، من هذه النقاط الثلاث يمكننا إيجاد متجهين اثنين، وليكونا $\vec{AB}$ & $\vec{AC}$

ولكننا نحتاج $\vec{N}$ فهنا نستطيع إيجاد $\vec{N}$ من خلال عمل vector prod. أو Cross prod. بين $\vec{AB}$ و $\vec{AC}$ وكما في التالي:

$$\vec{AB} = (7-2)i + (2-3)j + (1-5)K$$

$$\boxed{\vec{AB} = 5i - j - 4K}$$

$$\vec{AC} = (1-2)i + (1-3)j + (1-5)K$$

$$\boxed{\vec{AC} = -i - 2j - 4K}$$

$$\therefore \vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & K \\ 5 & -1 & -4 \\ -1 & -2 & -4 \end{vmatrix} = i\begin{vmatrix} -1 & -4 \\ -2 & -4 \end{vmatrix} - j\begin{vmatrix} 5 & -4 \\ -1 & -4 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ -1 & -2 \end{vmatrix}$$

$$\vec{N} = \boxed{-4i + 24j - 11K}$$

$\therefore$ To find the value of "d", we do a substitution of any point A, B, or C as follows;

$$ax + by + cz = d \quad ; \quad d = ax_0 + by_0 + cz_0$$

let $P_0 \Rightarrow C(1,1,1) \Rightarrow d = (-4 * 1) + (24 * 1) + (-11 * 1)$

$$\boxed{d = -9}$$

$$\therefore -4x + 24y - 11z = -9$$

Eqn of the plane

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Questions for discussions :-

① Find the parametric eqs for the lines for

a/ The line through the point $P(3,-4,-1)$ parallel to the vector $\vec{V} = i + j + k$

b/ The line through $P(-2,0,3)$ & $Q(3,5,-2)$

c/ The line through the origin parallel to vector $\vec{V} = 2j + k$

d/ The line through the point $(3,-2,1)$ parallel to the line $x = 1 + 2t$, $y = 2 - t$ & $z = 3t$

-- نهاية محاضرة " Vectors, Vectors in Space, Unit Vector, Scalar Product, Vector Product, line and plane equations المتجهات، المتجهات في الفضاء، وحدة المتجه، ضرب القيمة العددية، ضرب المتجه، معادلة الخط والمستوى" --