Complex Numbers :: الاعداد المركبة

About 300 years ago, equations such as $x^2+1=0$ & $x^2+2x+4=0$ had no solution, where $x=\mp\sqrt{-1}$. Until the scientist Carl Friedrich Gauss (1777-1855) came with what called a complex number.

These new numbers were of the form $a+ib$, where a & b are real & i satisfies the equation $i^2=-1$ i.e $i=\sqrt{-1}$

It is important to understand that the plus sign in $a+ib$ does not denote addition: rather $a+ib$ is a single number, not the sum of a and ib.

∴ The typical standard Cartesian Form of the complex number is

$$z=a+ib$$

a --- real part number
b --- imaginary part number.

$z=a+i0 \Longrightarrow$ purly real complex no.
$z=0+ib \Longrightarrow$ purly imaginary complex no.

Definitions:- تعاريف

① If $z = a + ib$, then $\bar{z} = a - ib$ is complex conjugate of z

② If $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$, then
$z_1 \mp z_2 = (a_1 \mp a_2) + i(b_1 \mp b_2)$

③ If $z_3 = a_3 + ib_3$ then;
* $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$ (تبادلي commutative)
* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, (تجميعي associative)
* $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
* $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (توزيعي distributive)

④ $z_1 * z_2 = (a_1 + ib_1)(a_2 + ib_2)$
$= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2$
$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

⑤ $z + (0 + i0) = z$; $(1 + i0) z = z$
i.e there are zero & unit complex numbers.

$0 = 0 + i0$
$1 = 1 + i0$

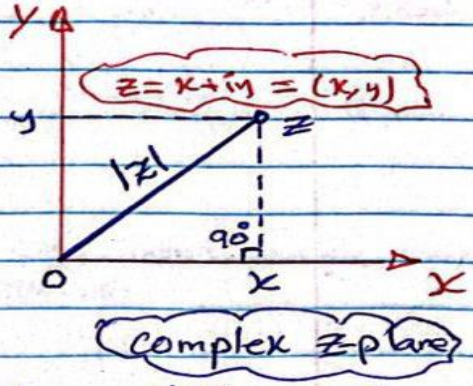
⑥ $z_1 = z_2$ if & only if $a_1 = a_2$ & $b_1 = b_2$

Complex plane : المستوى المركب

We can represent $z = a + ib$ as a point in a Cartesian a, b plane. And since Cartesian axes are more usually denoted by x and y rather than by a and b, let us write

$$z = x + iy = (x, y)$$

and represent z as a point in a so called "complex z plane"

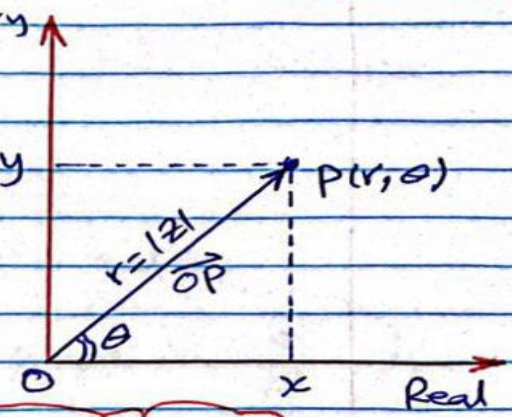


* The x- axis is called the "real axis"
* The y- axis is called the "imaginary axis"
* From The definition ②, the addition of the complex number satisfies the parallelogram law for the addition of vectors, so that the it is often convenient to think of complex numbers as vectors.
* The distance from the origin to the point z (i.e, the "length of the z vector") is called the modulus of z, and denoted as $|z|$

$$|z| = \sqrt{x^2 + y^2} \quad = \text{Modulus of } z$$

Polar Form of Complex number :=

The vector $\vec{OP}$ from the origin to $P(r,\theta)$ which is called polar form of complex number, where r & θ are called "polar coordinates"



$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$; θ = argument angle with x-axis

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ ← "Euler's Formula"

Ex(1) Proof that $e^{i\theta} = \cos\theta + i\sin\theta$?

Solution

By using a Taylor Series of $e^{i\theta}$ yields

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ Ans

Ex① proof that: $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

[Solution]

$e^{i\theta} = \cos\theta + i\sin\theta$ — ① Euler formula

$e^{-i\theta} = \cos\theta - i\sin\theta$ — ②

By adding eqs ① & ②, yields;

$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

② proof that: $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

[Solution]

By subtracting eqs ② From ①; yields,

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

More Definitions:-

If $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and,

$z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$ then,

⑦ $k \cdot z_1 = k(x_1 + iy_1) = kx_1 + ik y_1 = kr_1 e^{i\theta_1}$

⑧ $z_1 \times z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1r_2 e^{i(\theta_1+\theta_2)}$

⑨ $z_1 \cdot \bar{z}_1 = |z_1|^2$ & $z_2 \cdot \bar{z}_2 = |z_2|^2$

(10) $\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$

$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

(11) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}$; $r_2 \neq zero$

(12) $\bar{\bar{z}}_1 = z_1$ & $\bar{\bar{z}}_2 = z_2$

Ex) Evaluate $\frac{2+i}{3-4i}$

Solution

$\frac{2+i}{3-4i} * \frac{3+4i}{3+4i} = \frac{(6-4)+(8+3)i}{9+16} = \frac{2}{25} + \frac{11}{25}i$

Ex) let $z_1 = 2+i3$ & $z_2 = 4+i$

Find; ① z_1+z_2
② z_1-z_2
③ $z_1 \cdot z_2$

Solution

① $z_1+z_2 = (2+i3)+(4+i) = (2+4)+(3+1)i = \boxed{6+i4}$

② $z_1-z_2 = (2+i3)-(4+i) = (2-4)+(3-1)i = \boxed{-2+i2}$

③ $z_1 * z_2 = (2+i3)*(4+i) = 2*4+i2+3*4i+3*1*i*i$
$= 8+i2+i12+3i^2 = 8-3+i14$
$= \boxed{5+i14}$; $\boxed{i^2=-1}$

Ex/ put the complex number $(1-i\sqrt{3})$ in the polar form?

Solution

$$z = 1-i\sqrt{3} = x+iy = |z|e^{i\theta} = re^{i\theta}$$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{1^2+(-\sqrt{3})^2} = \sqrt{4} = \boxed{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{1} = \tan^{-1}(-\sqrt{3}) = \boxed{-60^\circ}$$

$$= \boxed{300^\circ} = \boxed{\frac{5\pi}{3}}$$

$$\therefore z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

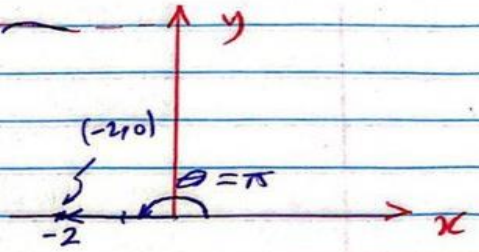
$$= \boxed{2e^{i\frac{5\pi}{3}} = 2(\cos 300 + i\sin 300)} \quad \text{Ans}$$

Note

$$\boxed{\ln re^{i\theta} = \ln r + i\theta}$$

Ex/ Find $\ln(-2)$?

Solution



$$\ln -2 = \ln(-2+i0)$$

$$r = \sqrt{(-2)^2+(0)^2} = \sqrt{4} = \boxed{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{-2} = \tan^{-1}0 = \boxed{0}$$

Here we need to correct the angle by add π

$$\therefore \boxed{\theta = \pi}$$

$$\therefore \boxed{\ln(-2) = \ln 2 + i\pi} \quad \text{Ans}$$

Ex) Find x, y if $(3+i4)^2 - 2(x-iy) = x+iy$

Solution:

$9 + i\times 2\times 3\times 4 + (i4)^2 - 2x + i2y = x+iy$

$9 + i24 + i^2 16 - 2x + i2y = x+iy$ ($i^2 = -1$)

$9 + i24 - 16 - 2x + i2y = x+iy$

$-7 - 2x + i(24+2y) = x+iy$

* From Def # (6)

$\therefore -7 - 2x = x \longrightarrow -7 = 3x \longrightarrow \boxed{x = -7/3}$ Ans

$\&\ 24 + 2y = y \longrightarrow \boxed{y = -24}$ Ans

De Moivre's Theorem :- نظرية ديموفر

If n is any integer (positive, negative, or zero), and z is a complex number in a polar form ; then,

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

This equation is well known as "De Moivre's formula"

Note!

In complex plane θ is not uniquely determined at any given point z, so, if we put $\theta = \theta_0 + 2\pi k$ into the power polar form, gives

$$z^n = r^n e^{in(\theta_0 + 2\pi k)} = r^n e^{in\theta_0} e^{i2\pi kn}$$

Then, $e^{i2\pi kn} = \cos(2\pi kn) + i \sin(2\pi kn) = 1 + i0$

$= 1$ for all integer K.

So, $z^n = r^n e^{in\theta_0}$ and $\boxed{\theta_0 = \theta}$

Ex! compute $(1+i)^3$?

Solution

$\boxed{n=3}$

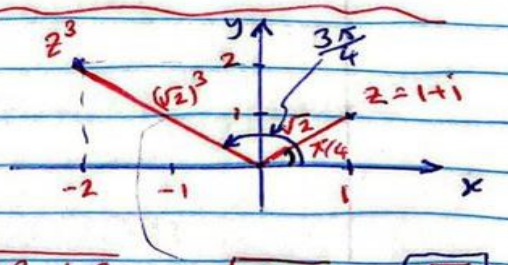
$z = 1 + i = a + ib$

$a = 1$, $b = 1$ $\Longrightarrow$ $|z| = r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \boxed{\sqrt{2}}$

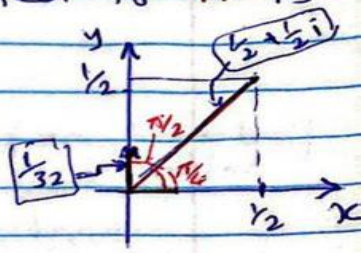
$\theta = \tan^{-1}\frac{b}{a} \Longrightarrow \theta = \tan^{-1}\frac{1}{1} = \boxed{45^0} = \boxed{\frac{\pi}{4}}$

$z^n = r^n(\cos n\theta + i \sin n\theta) = (\sqrt{2})^3(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$

$= 2^{\frac{3}{2}}(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \boxed{-2+2i}$



We can also solve the previous example by,-
$(1+i)^3 = (1+i)(1+i)(1+i) = (1+i)(2i) = \boxed{2i-2}$, but it is gonna be quite exhausted if n is large.



Ex) Compute $(\frac{1}{2}+i\frac{1}{2})^{10}$?

Solution

$\boxed{n=10}$; $a=\frac{1}{2}$, $b=\frac{1}{2}$

$r = \sqrt{a^2+b^2} = \sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2} = \boxed{\frac{1}{\sqrt{2}}}$

$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1/2}{1/2} = \boxed{45^0} = \boxed{\frac{\pi}{4}}$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$= (\frac{1}{\sqrt{2}})^{10}(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi)$

$= \frac{1}{32}(0+i) = \boxed{\frac{1}{32}i} \leftarrow \boxed{\theta = \frac{\pi}{2}}$

Fractional Powers :-

If we consider the function $z^{1/n}$, which called the n^{th} root of z, we write;

$z^{1/n} = (r e^{i(\theta_0+2\pi k)})^{1/n} = r^{1/n} e^{i(\frac{\theta_0+2\pi k}{n})}$

$$\boxed{z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta_0+2\pi k}{n}\right) + i\sin\left(\frac{\theta_0+2\pi k}{n}\right)\right] = F_k}$$

$k = 1, 2, 3, \dots, n-1$

Ex) Find the value of $(-1+i)^{1/3}$?

Solution

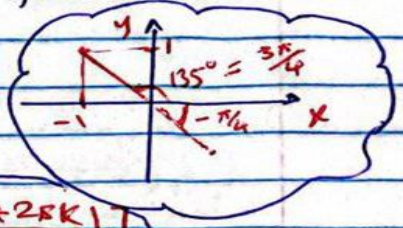
$n=3$, $K=0,1,2$

$r=\sqrt{a^2+b^2}=\sqrt{(-1)^2+(1)^2}=\boxed{\sqrt{2}}$

$\theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{1}{-1}=\boxed{-45^\circ}$ (need to be corrected)

the complex no. is in 2nd quarter!

$\therefore \theta=180-45=\boxed{135^\circ}=\boxed{\frac{3\pi}{4}}$



$$z^{1/n}=r^{1/n}\left[\cos\left(\frac{\theta+2\pi K}{n}\right)+i\sin\left(\frac{\theta+2\pi K}{n}\right)\right]$$

$$(-1+i)^{1/3}=(\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi K}{3}\right)+i\sin\left(\frac{\frac{3\pi}{4}+2\pi K}{3}\right)\right]$$

① At $K=0 \longrightarrow (1+i)^{1/3}=(\sqrt{2})^{1/3}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$

$$=\boxed{(\sqrt{2})^{1/3}\left[\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right]}$$

② At $K=1 \longrightarrow (-1+i)^{1/3}=(\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right)+i\sin\left(\frac{\frac{3\pi}{4}+2\pi}{3}\right)\right]$

$$=(\sqrt{2})^{1/3}\left[\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right]$$

$$=\boxed{(\sqrt{2})^{1/3}[-0.966+i\,0.2588]}$$

③ At $K=2 \longrightarrow (-1+i)^{1/3}=(\sqrt{2})^{1/3}\left[\cos\left(\frac{\frac{3\pi}{4}+4\pi}{3}\right)+i\sin\left(\frac{\frac{3\pi}{4}+4\pi}{3}\right)\right]$

$$=(\sqrt{2})^{1/3}\left[\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right]$$

$$=\boxed{(\sqrt{2})^{1/3}[0.2588-0.966\,i]}$$

Ex/ Find the sixth roots of $z=-8$?

Solution:

$z=-8+0i \quad \leftarrow a+ib \quad ; \quad a=-8, \; b=0$

$r=\sqrt{a^2+b^2} = \sqrt{(-8)^2+0^2} = \boxed{8}$

$\theta=\tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{-8} = \tan^{-1}0 = 0$ (need correction)

$\theta=\pi+2\pi(0) = \boxed{\pi}$

$n=6$

$k=0,1,2,3,4,5$

$$z^{1/6}=r^{1/6}\left[\cos\left(\frac{\theta+2\pi k}{6}\right)+i\sin\left(\frac{\theta+2\pi k}{6}\right)\right]$$

① at $k=0 \Rightarrow F_0 = 8^{1/6}\left[\cos\left(\frac{\pi+2\pi(0)}{6}\right)+i\sin\left(\frac{\pi+2\pi(0)}{6}\right)\right]$

$= 8^{1/6}\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$

$= \boxed{\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)}$

② at $k=1 \Rightarrow F_1 = 8^{1/6}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = \boxed{\sqrt{2}\,i}$

③ at $k=2 \Rightarrow F_2 = 8^{1/6}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right)}$

④ at $k=3 \Rightarrow F_3 = 8^{1/6}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{-\sqrt{3}}{2}-\frac{1}{2}i\right)}$

⑤ at $k=4 \Rightarrow F_4 = 8^{1/6}\left(\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}\right) = \boxed{-\sqrt{2}\,i}$

⑥ at $k=5 \Rightarrow F_5 = 8^{1/6}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right) = \boxed{\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)}$

Ex)

Find parametric equations for line through the points $P(-3,2,-3)$ & $Q(1,-1,4)$?

(Solution)

$$\vec{PQ} = (1-(-3))\,i + (-1-2)\,j + (4-(-3))\,k$$

$$\boxed{\vec{PQ} = 4i - 3j + 7k} = \boxed{Ai + Bj + Ck}$$

* let $(x_0, y_0, z_0) = (-3, 2, -3)$

$$\therefore \boxed{x = 4t - 3} \; ; \; \boxed{y = -3t + 2} \; ; \; \boxed{z = 7t - 3}$$

or;

* let $(x_0, y_0, z_0) = (1, -1, 4)$

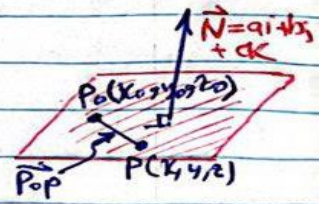
$$\therefore \boxed{x = 4t + 1} \; ; \; \boxed{y = -3t + 1} \; ; \; \boxed{z = 7t + 4}$$

Equation of the plane : معادلة المستوي

To find the eqn of the plane that passes through the point $P_0(x_0, y_0, z_0)$ & it's normal vector is $\vec{N} = ai + bj + ck$

Let $p(x, y, z)$ be any point in the plane

$$\vec{P_0P} = (x - x_0)\,i + (y - y_0)\,j + (z - z_0)\,k$$



$$\vec{P_0P} \perp N \Longrightarrow \vec{P_0P} \cdot \vec{N} = 0$$

$$\Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Longrightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$\boxed{ax + by + cz = d}$$ ← Equation of the plane

$$f'(z)=\lim_{\Delta y\to 0}\frac{[u(x,y+\Delta y)+iv(x,y+\Delta y)]-[u(x,y)+iv(x,y)]}{i\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{[u(x,y+\Delta y)-u(x,y)}{i\Delta y}+\frac{i\lim_{\Delta y\to 0}[v(x,y+\Delta y)+v(x,y)]}{i\Delta y}$$

$$f'(z)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \quad (2)$$

Thus, the eq-s (1) & (2) must be equal to have $f'(z)$ exist at z, so,

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

Ex) Examine weither or not the eq- $f(z)=|z|^2$ differentiable ??

Solution

$f(z)=|z|^2$; $z=x+iy \longrightarrow |z|=\sqrt{x^2+y^2}$

$\therefore f(z)=(x^2+y^2)+i0$

$\therefore u=x^2+y^2 \longrightarrow \frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=2y$

$v=0 \longrightarrow \frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=0$

$\therefore$ just at the origin $(0,0)$ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

So, this eq- is differentiable only at $z=0$

$$f'(z)=\lim_{\Delta y\to 0}\frac{[u(x,y+\Delta y)+iv(x,y+\Delta y)]-[u(x,y)+iv(x,y)]}{i\Delta y}$$

$$=\lim_{\Delta y\to 0}\frac{[u(x,y+\Delta y)-u(x,y)}{i\Delta y}+\frac{i\lim_{\Delta y\to 0}[v(x,y+\Delta y)+v(x,y)]}{i\Delta y}$$

$$f'(z)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \quad (2)$$

Thus, the eqs ① & ② must be equal to have $f'(z)$ exist at z, so,

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

Ex) Examine weither or not the eq $f(z)=|z|^2$ differentiable ??

Solution

$f(z)=|z|^2$; $z=x+iy \longrightarrow |z|=\sqrt{x^2+y^2}$

$\therefore f(z)=(x^2+y^2)+i0$

$\therefore u=x^2+y^2 \longrightarrow \frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=2y$

$v=0 \longrightarrow \frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=0$

$\therefore$ just at the origin $(0,0)$ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

So, this eq is differentiable only at $z=0$

Ex) Show that $f(z) = z^2$ satisfies Cauchy–Riemann equations, & find the derivative in terms of the partial derivative?

Solution)

$f(z) = z^2$; $z = x + iy$

$\therefore f(z) = (x+iy)^2 = (x^2 - y^2) + i\,2xy$

So,

$u(x,y) = x^2 - y^2 \Longrightarrow \frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$

$v(x,y) = 2xy \Longrightarrow \frac{\partial v}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = 2x$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$

∴ The Cauchy–Riemann eq-s are satisfied & the Fun is differentiable.

To find the derivative then;

$$f'(z) = u_x + i v_x$$

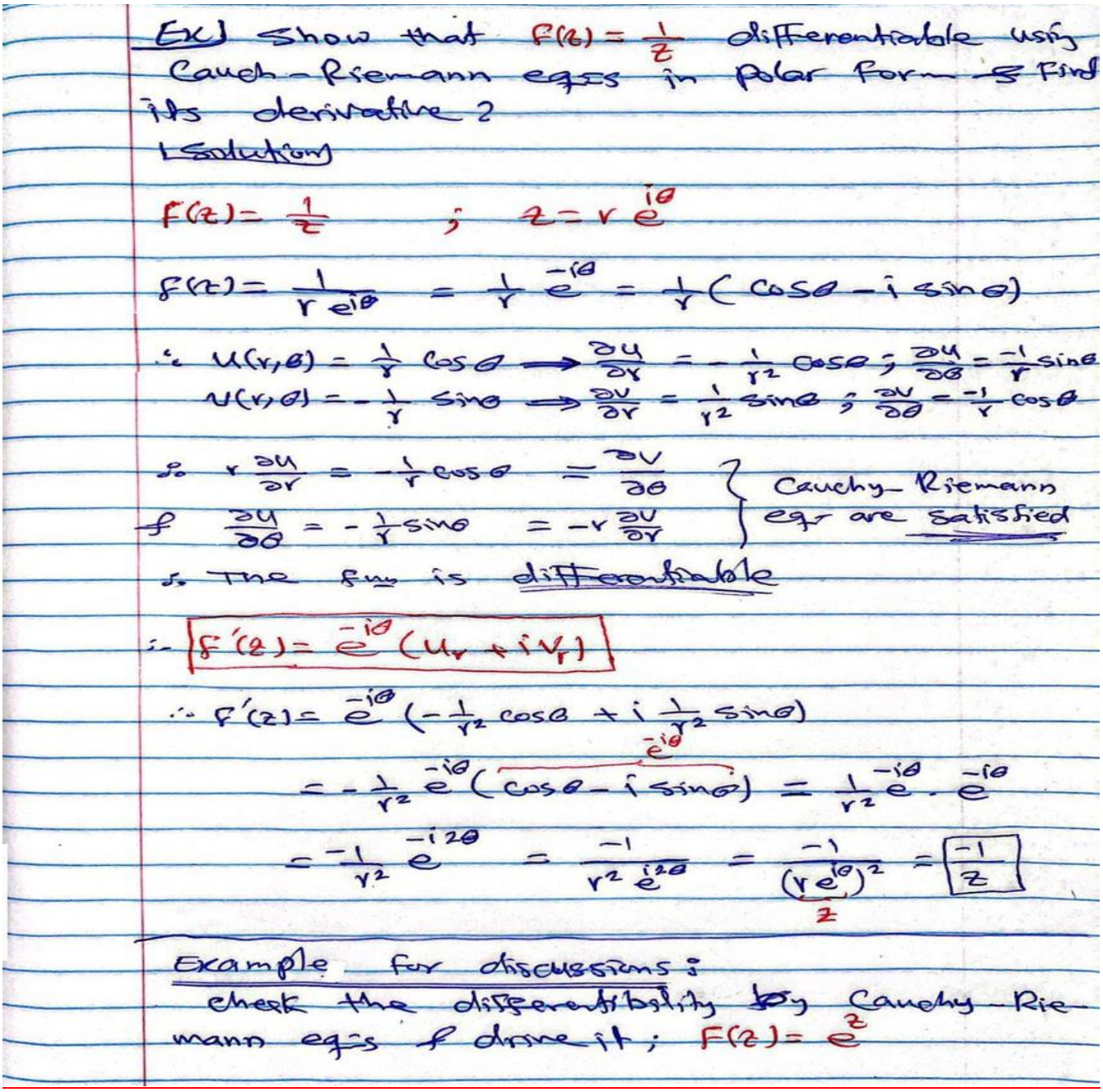
$$f'(z) = 2x + i2y = 2\overbrace{(x+iy)}^{z} = \boxed{2z}$$

Cauch. Riemann Eq-s in Polar Form

معادلة كوشي – ريمان بالإحداثيات القطبية

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$



Ex) Show that $F(z)=\frac{1}{z}$ differentiable using Cauchy-Riemann eqs in polar form & Find its derivative ?

Solution

$F(z)=\frac{1}{z}$; $z=re^{i\theta}$

$f(z)=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}=\frac{1}{r}(\cos\theta-i\sin\theta)$

$\therefore u(r,\theta)=\frac{1}{r}\cos\theta \rightarrow \frac{\partial u}{\partial r}=-\frac{1}{r^2}\cos\theta$; $\frac{\partial u}{\partial\theta}=\frac{-1}{r}\sin\theta$

$v(r,\theta)=-\frac{1}{r}\sin\theta \rightarrow \frac{\partial v}{\partial r}=\frac{1}{r^2}\sin\theta$; $\frac{\partial v}{\partial\theta}=\frac{-1}{r}\cos\theta$

$\therefore r\frac{\partial u}{\partial r}=-\frac{1}{r}\cos\theta=\frac{\partial v}{\partial\theta}$

$\&\ \frac{\partial u}{\partial\theta}=-\frac{1}{r}\sin\theta=-r\frac{\partial v}{\partial r}$

} Cauchy-Riemann eqs are satisfied

$\therefore$ The fun is differentiable

$\therefore \boxed{f'(z)=e^{-i\theta}(u_r+iv_r)}$

$\therefore f'(z)=e^{-i\theta}(-\frac{1}{r^2}\cos\theta+i\frac{1}{r^2}\sin\theta)$

$=-\frac{1}{r^2}e^{-i\theta}(\cos\theta-i\sin\theta)=\frac{1}{r^2}e^{-i\theta}\cdot e^{-i\theta}$

$=\frac{-1}{r^2}e^{-i2\theta}=\frac{-1}{r^2e^{i2\theta}}=\frac{-1}{(re^{i\theta})^2}=\boxed{\frac{-1}{z^2}}$

Example for discussions :
check the differentiability by Cauchy-Riemann eqs & prove it ; $F(z)=e^z$

Complex numbers, Plar Formulam Euler equ, Power and Roots, Complex Func, Cautchy-Reiman Eqs -- نهاية محاضرة " الاعداد المركبة، الشكل القطبي، معادلو أويلر، قوى وجذور الاعداد المركبة، دالة العدد المركب، معادلات كوشي ريمن "--

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