



Complex Numbers :: الأعداد المركبة

About 300 years ago, equations such as $x^2+1=0$ or $x^2+2x+4=0$ had no solution, where $x \in \mathbb{R}$. Until the scientist Carl Friedrich Gauss (1777-1855) came with what called a complex number.

These new numbers were of the form $a+ib$, where a & b are real & i satisfies the equation $i^2 = -1$ i.e. $i = \sqrt{-1}$.

It is important to understand that the plus sign in $a+ib$ does not denote addition; rather $a+ib$ is a single number, not the sum of a and ib .

The typical standard Cartesian Form of the complex number is

$$z = a + ib$$

a -- real part number

b -- imaginary part number.

$z = a + i0 \Rightarrow$ purely real complex no.

$z = 0 + ib \Rightarrow$ purely imaginary complex no.



Definitions:

تعريف

① If $z = a + ib$, then $\bar{z} = a - ib$ is complex conjugate of z

② If $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$, then
 $z_1 \mp z_2 = (a_1 \mp a_2) + i(b_1 \mp b_2)$

③ If $z_3 = a_3 + ib_3$ then;

$$* z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$$

تبادلي
(commutative)

$$* (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$

علاقه
(associative)

$$* (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$* z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

توزيعي
(distributive)

$$\begin{aligned} \textcircled{4} z_1 * z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= a_1 a_2 + ia_1 b_2 + ia_2 b_1 + i^2 b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \end{aligned}$$

⑤ $z + (0 + i0) = z$; $(1 + i0)z = z$
i.e there are zero & unit complex numbers.

$$\begin{aligned} 0 &= 0 + i0 \\ 1 &= 1 + i0 \end{aligned}$$

⑥ $z_1 = z_2$ if & only if $a_1 = a_2$ & $b_1 = b_2$



Complex plane :

المستوى المركب

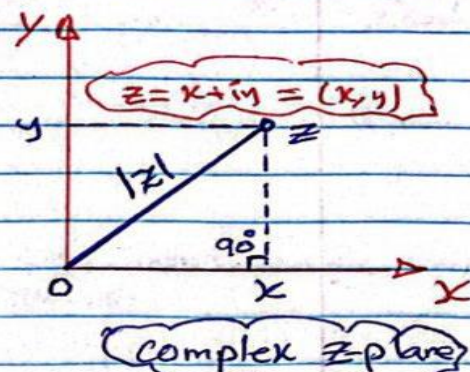
We can represent $z = a + ib$ as a point in a Cartesian a, b plane. And since Cartesian axes are more usually denoted by x and y rather than by a and b , let us write

$$z = x + iy = (x, y)$$

and represent z as a point in a so called "Complex z plane"

* The x -axis is called the "real axis"

* The y -axis is called the "imaginary axis"



* From The definition ②, the addition of the complex number satisfies the parallelogram law for the addition of vectors, so that the it is often convenient to think of complex numbers as vectors.

* The distance from the origin to the point z (i.e., the "length of the z vector") is called the modulus of z , and denoted as $|z|$

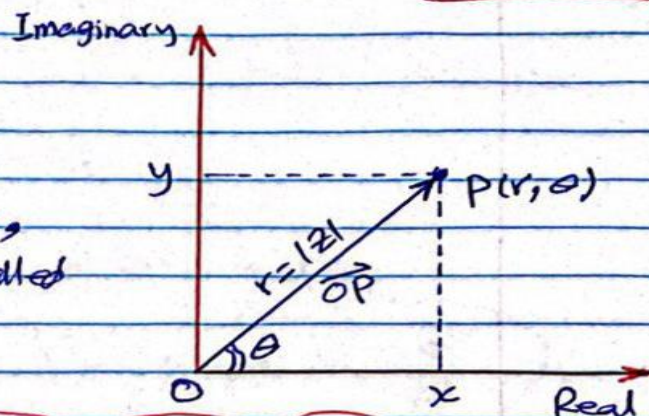
$$|z| = \sqrt{x^2 + y^2} = \text{Modulus of } z$$



Polar Form of Complex number

العدد المركب بالصيغة القطبية
أو الصيغة الأسية

The vector $\vec{OP}$ from the origin to $P(r, \theta)$ which is called polar form of complex number, where r & θ are called "polar coordinates"



$$Z = x + iy = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

; θ = argument angle with x -axis

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

"Euler's Formula"

Ex1) Prove that $e^{i\theta} = \cos \theta + i \sin \theta$?

Solution

By using a Taylor series of $e^{i\theta}$ yields

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!} (i\theta)^2 + \frac{1}{3!} (i\theta)^3 + \frac{1}{4!} (i\theta)^4 + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \end{aligned}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Ans



Ex 1) proof that: $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

[solution]

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{--- Euler's Formula} \quad \text{--- (1)}$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \quad \text{--- (2)}$$

By adding eqs 1 & 2, yields;

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

2) proof that: $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

[solution]

By subtracting eqs 2 from 1; yields,

$$e^{i\theta} - e^{-i\theta} = 2i \sin \theta \Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

More Definitions

$$\text{If } z_1 = x_1 + iy_1 = r_1 e^{i\theta_1} \quad \text{and,}$$

$$z_2 = x_2 + iy_2 = r_2 e^{i\theta_2} \quad \text{then,}$$

$$\textcircled{7} k \cdot z_1 = k(x_1 + iy_1) = kx_1 + iky_1 = kr_1 e^{i\theta_1}$$

$$\textcircled{8} z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\textcircled{9} z_1 \cdot \bar{z}_1 = |z_1|^2 \quad \& \quad z_2 \cdot \bar{z}_2 = |z_2|^2$$



$$\textcircled{10} \frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$
$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\textcircled{11} \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \quad ; \quad r_2 \neq 0$$

$$\textcircled{12} \overline{\overline{z_1}} = z_1 \quad \Leftarrow \quad \overline{\overline{z_2}} = z_2$$

Ex Evaluate $\frac{2+i}{3-4i}$
Solution

$$\frac{2+i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{(6-4) + (8+3)i}{9+16} = \frac{2}{25} + \frac{11}{25}i$$

Ex let $z_1 = 2+i3 \quad \Leftarrow \quad z_2 = 4+i$
Find: $\textcircled{1} z_1 + z_2$
 $\textcircled{2} z_1 - z_2$
 $\textcircled{3} z_1 \cdot z_2$

Solution

$$\textcircled{1} z_1 + z_2 = (2+i3) + (4+i) = (2+4) + (3+1)i = \boxed{6+i4}$$

$$\textcircled{2} z_1 - z_2 = (2+i3) - (4+i) = (2-4) + (3-1)i = \boxed{-2+i2}$$

$$\textcircled{3} z_1 \cdot z_2 = (2+i3) \times (4+i) = 2 \times 4 + i2 + 3 \times 4i + 3 \times i \times i$$
$$= 8 + i2 + i12 + 3i^2 = 8 - 3 + i14$$
$$= \boxed{5+i14} \quad ; \quad \boxed{i^2 = -1}$$

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Ex put the Complex number $(1 - i\sqrt{3})$ in the Polar Form?

Solution

$$Z = 1 - i\sqrt{3} = x + iy = |Z| e^{i\theta} = r e^{i\theta}$$

$$|Z| = r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = \boxed{2}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-\sqrt{3}}{1} = \tan^{-1}(-\sqrt{3}) = \boxed{-60^\circ} \\ = \boxed{360^\circ} = \boxed{\frac{5\pi}{3}}$$

$$\therefore Z = r e^{i\theta} = r (\cos \theta + i \sin \theta) \\ = \boxed{2 e^{i\frac{5\pi}{3}}} = \boxed{2 (\cos 300 + i \sin 300)} \quad \text{Ans}$$

Note

$$\ln r e^{i\theta} = \ln r + i\theta$$

Ex Find $\ln(-2)$?

Solution

$$\ln -2 = \ln(-2 + i0)$$

$$r = \sqrt{(-2)^2 + (0)^2} = \sqrt{4} = \boxed{2}$$

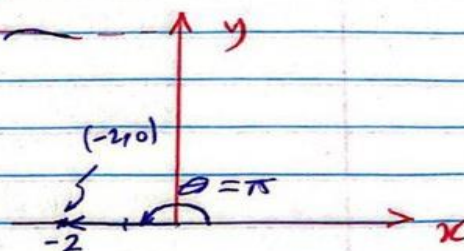
$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{0}{-2} = \tan^{-1} 0 = \boxed{0}$$

Here we need to correct the angle by add π

$$\therefore \theta = \pi$$

$$\therefore \ln(-2) = \ln 2 + i\pi$$

Ans



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Ex) Find x, y if $(3 + i4)^2 - 2(x + iy) = x + iy$

Solution

$$9 + i \times 2 \times 3 \times 4 + (i4)^2 - 2x + i2y = x + iy$$

$$9 + i24 + i^2 16 - 2x + i2y = x + iy$$

$$9 + i24 - 16 - 2x + i2y = x + iy$$

$$-7 - 2x + i(24 + 2y) = x + iy$$

* From Def # ⑥

$$\therefore -7 - 2x = x \rightarrow -7 = 3x \rightarrow \boxed{x = -\frac{7}{3}} \text{ Ans}$$

$$\S 24 + 2y = y \rightarrow \boxed{y = -24} \text{ Ans}$$



De Moivre's Theorem

نظريّة دي مويفر

If n is any integer (positive, negative, or zero), and z is a complex number in a polar form; then,

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

This equation is well known as "De Moivre's formula"

Note

In complex plane θ is not uniquely determined at any given point z , so, if we put $\theta = \theta_0 + 2\pi k$ into the power polar form, gives

$$z^n = r^n e^{in(\theta_0 + 2\pi k)} = r^n e^{in\theta_0} e^{i2\pi kn}$$

$$\text{Then, } e^{i2\pi kn} = \cos(2\pi kn) + i \sin(2\pi kn) = 1 + i0 = 1 \text{ for all integer } k$$

$$\text{So, } z^n = r^n e^{in\theta_0} \text{ and } \boxed{\theta_0 = \theta}$$

Ex compute $(1+i)^3$?

Solution

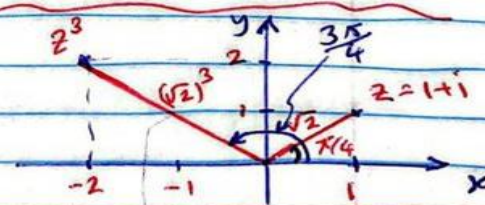
$$\boxed{n=3}$$

$$z = 1+i = a+ib$$

$$\left. \begin{array}{l} a=1 \\ b=1 \end{array} \right\} \Rightarrow |z| = r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \boxed{\sqrt{2}}$$

$$\theta = \tan^{-1} \frac{b}{a} \Rightarrow \theta = \tan^{-1} 1 = \boxed{45^\circ} = \boxed{\frac{\pi}{4}}$$

$$\begin{aligned} z^n &= r^n (\cos n\theta + i \sin n\theta) = (\sqrt{2})^3 (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) \\ &= 2^{\frac{3}{2}} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \boxed{-2 + 2i} \end{aligned}$$



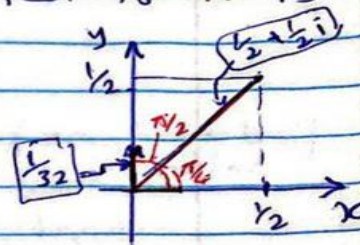


We can also solve the previous example by,
 $(1+i)^3 = (1+i)(1+i)(1+i) = (1+i)(2i) = 2i - 2$, but
it is gonna be quite exhausted if n is
large.

Ex] Compute $(\frac{1}{2} + i\frac{1}{2})^{10}$?

Solution

$n=10$; $a = \frac{1}{2}$, $b = \frac{1}{2}$



$$r = \sqrt{a^2 + b^2} = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1/2}{1/2} = 45^\circ = \frac{\pi}{4}$$

$$Z^n = r^n (\cos n\theta + i \sin n\theta)$$

$$= (\frac{1}{\sqrt{2}})^{10} (\cos \frac{10}{4} \pi + i \sin \frac{10}{4} \pi)$$

$$= \frac{1}{32} (0 + i) = \frac{1}{32} i \leftarrow \theta = \frac{\pi}{2}$$

Fractional powers :

If we consider the function $Z^{1/n}$, which
called the n^{th} root of Z , we write;

$$Z^{1/n} = (r e^{i(\theta_0 + 2\pi k)})^{1/n} = r^{1/n} e^{i(\frac{\theta_0 + 2\pi k}{n})}$$

$$Z^{1/n} = r^{1/n} [\cos(\frac{\theta_0 + 2\pi k}{n}) + i \sin(\frac{\theta_0 + 2\pi k}{n})] = F_k$$

$$k = 1, 2, 3, \dots, n-1$$



Ex) Find the value of $(-1+i)^{1/3}$?

Solution

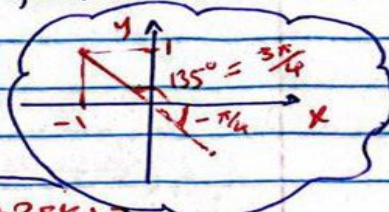
$$n=3, K=0,1,2$$

$$r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(1)^2} = \boxed{\sqrt{2}}$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1}{-1} = \boxed{-45^\circ} \text{ (need to be corrected)}$$

the complex no. is in 2nd quarter?

$$\therefore \theta = 180 - 45 = \boxed{135^\circ} = \boxed{\frac{3\pi}{4}}$$



$$z^{1/n} = r^{1/n} \left[\cos\left(\frac{\theta + 2\pi K}{n}\right) + i \sin\left(\frac{\theta + 2\pi K}{n}\right) \right]$$

$$(-1+i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos\left(\frac{3\pi/4 + 2\pi K}{3}\right) + i \sin\left(\frac{3\pi/4 + 2\pi K}{3}\right) \right]$$

$$\textcircled{1} \text{ At } K=0 \rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] \\ = \boxed{(\sqrt{2})^{1/3} \left[\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right]}$$

$$\textcircled{2} \text{ At } K=1 \rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos\left(\frac{3\pi/4 + 2\pi}{3}\right) + i \sin\left(\frac{3\pi/4 + 2\pi}{3}\right) \right] \\ = (\sqrt{2})^{1/3} \left[\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right] \\ = \boxed{(\sqrt{2})^{1/3} [-0.966 + i 0.2588]}$$

$$\textcircled{3} \text{ At } K=2 \rightarrow (-1+i)^{1/3} = (\sqrt{2})^{1/3} \left[\cos\left(\frac{3\pi/4 + 4\pi}{3}\right) + i \sin\left(\frac{3\pi/4 + 4\pi}{3}\right) \right] \\ = (\sqrt{2})^{1/3} \left[\cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12} \right] \\ = \boxed{(\sqrt{2})^{1/3} [0.2588 - 0.966 i]}$$



Ex: Find the sixth roots of $z = -8$?

Solution

$$z = -8 + 0i = a + ib \quad ; \quad a = -8, \quad b = 0$$

$$r = \sqrt{a^2 + b^2} = \sqrt{(-8)^2 + 0^2} = \boxed{8}$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{0}{-8} = \tan^{-1} 0 = 0 \quad (\text{need correction})$$

$$\theta = \pi + 2\pi n = \boxed{\pi}$$

$$n = 6$$

$$k = 0, 1, 2, 3, 4, 5$$

$$z^{1/6} = \sqrt[6]{8} \left[\cos\left(\frac{\theta + 2\pi k}{6}\right) + i \sin\left(\frac{\theta + 2\pi k}{6}\right) \right]$$

$$\begin{aligned} \textcircled{1} \text{ at } k=0 &\Rightarrow F_0 = \sqrt[6]{8} \left[\cos\left(\frac{\pi + 2\pi(0)}{6}\right) + i \sin\left(\frac{\pi + 2\pi(0)}{6}\right) \right] \\ &= \sqrt[6]{8} [\cos \pi + i \sin \pi] \\ &= \sqrt[6]{8} (-1 + i \cdot 0) = \boxed{-\sqrt[6]{8}} \end{aligned}$$

$$\textcircled{2} \text{ at } k=1 \Rightarrow F_1 = \sqrt[6]{8} (\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = \boxed{\sqrt[6]{8} i}$$

$$\textcircled{3} \text{ at } k=2 \Rightarrow F_2 = \sqrt[6]{8} (\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}) = \boxed{-\sqrt[6]{8} i}$$

$$\textcircled{4} \text{ at } k=3 \Rightarrow F_3 = \sqrt[6]{8} (\cos \pi + i \sin \pi) = \boxed{-\sqrt[6]{8}}$$

$$\textcircled{5} \text{ at } k=4 \Rightarrow F_4 = \sqrt[6]{8} (\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}) = \boxed{\sqrt[6]{8} i}$$

$$\textcircled{6} \text{ at } k=5 \Rightarrow F_5 = \sqrt[6]{8} (\cos \frac{7\pi}{2} + i \sin \frac{7\pi}{2}) = \boxed{-\sqrt[6]{8} i}$$



Ex1

Find parametric equations for line through the points $P(-3, 2, -3)$ & $Q(1, -1, 4)$?

(solution)

$$\vec{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k}$$

$$\vec{PQ} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} = [A\mathbf{i} + B\mathbf{j} + C\mathbf{k}]$$

* let $(x_0, y_0, z_0) = (-3, 2, -3)$

$$\therefore \boxed{x = 4t - 3} \quad ; \quad \boxed{y = -3t + 2} \quad ; \quad \boxed{z = 7t - 3}$$

or,

* let $(x_0, y_0, z_0) = (1, -1, 4)$

$$\therefore \boxed{x = 4t + 1} \quad ; \quad \boxed{y = -3t + 1} \quad ; \quad \boxed{z = 7t + 4}$$

Equation of the plane : CS geometry notes

To find the eqn of the plane that passes through the point $P_0(x_0, y_0, z_0)$ & its normal vector is $\vec{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Let $P(x, y, z)$ be any point in the plane

$$\vec{P_0P} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

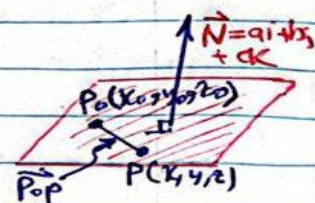
$$\vec{P_0P} \perp \vec{N} \Rightarrow \vec{P_0P} \cdot \vec{N} = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Rightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$\boxed{ax + by + cz = d} \leftarrow \text{Equation of the plane}$$





$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) + i v(x, y+\Delta y)] - [u(x, y) + i v(x, y)]}{i \Delta y}$$
$$= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) - u(x, y)]}{i \Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{[v(x, y+\Delta y) - v(x, y)]}{\Delta y}$$

$$f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad (2)$$

Thus, the eqs ① & ② must be equal to have $f'(z)$ exist at z , so,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Ex) Examine whether or not the eq = $f(z) = |z|^2$ differentiable?

Solution

$$f(z) = |z|^2 \quad ; \quad z = x + iy \rightarrow |z| = \sqrt{x^2 + y^2}$$

$$\therefore f(z) = (x^2 + y^2) + i0$$

$$\therefore u = x^2 + y^2 \rightarrow \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$
$$v = 0 \rightarrow \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\therefore \text{just at the origin } (0,0) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, this eq is differentiable only at $z=0$



$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) + i v(x, y+\Delta y)] - [u(x, y) + i v(x, y)]}{i \Delta y}$$
$$= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) - u(x, y)]}{i \Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{[v(x, y+\Delta y) - v(x, y)]}{\Delta y}$$

$$f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad (2)$$

thus, the eqs ① & ② must be equal to have $f'(z)$ exist at z , so,

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$$\therefore u = x^2 + y^2 \rightarrow \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$
$$v = 0 \rightarrow \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\therefore \text{just at the origin } (0,0) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so, this eqs is differentiable only at $z=0$



Ex) Show that $f(z) = z^2$ satisfies Cauchy-Riemann equations, & find the derivative in terms of the partial derivative?

Solution

$$f(z) = z^2 ; z = x + iy$$

$$\therefore f(z) = (x + iy)^2 = (x^2 - y^2) + i 2xy$$

So,

$$u(x, y) = x^2 - y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x ; \frac{\partial u}{\partial y} = -2y$$

$$v(x, y) = 2xy \Rightarrow \frac{\partial v}{\partial x} = 2y ; \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y$$

$\therefore$ The Cauchy-Riemann eq-s are satisfied & the fcn is differentiable.

To find the derivative then;

$$f'(z) = u_x + i v_x$$

$$f'(z) = 2x + i 2y = 2(x + iy) = \boxed{2z}$$

Cauchy-Riemann Eq-s in Polar Form so
نكتب المعادلات في الصورة القطبية

$$\begin{aligned} r \frac{\partial u}{\partial r} &= \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r \frac{\partial v}{\partial r} \end{aligned}$$



Ex) Show that $F(z) = \frac{1}{z}$ differentiable using Cauchy-Riemann eqs in polar form & find its derivative?

Solution

$$F(z) = \frac{1}{z} \quad ; \quad z = r e^{i\theta}$$

$$F(z) = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r} (\cos\theta - i \sin\theta)$$

$$\therefore u(r, \theta) = \frac{1}{r} \cos\theta \rightarrow \frac{\partial u}{\partial r} = -\frac{1}{r^2} \cos\theta ; \frac{\partial u}{\partial \theta} = -\frac{1}{r} \sin\theta$$

$$v(r, \theta) = -\frac{1}{r} \sin\theta \rightarrow \frac{\partial v}{\partial r} = \frac{1}{r^2} \sin\theta ; \frac{\partial v}{\partial \theta} = -\frac{1}{r} \cos\theta$$

$$\therefore \left. \begin{aligned} r \frac{\partial u}{\partial r} &= -\frac{1}{r} \cos\theta = \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -\frac{1}{r} \sin\theta = -r \frac{\partial v}{\partial r} \end{aligned} \right\} \text{Cauchy-Riemann eqs are satisfied}$$

$\therefore$ The fcn is differentiable

$$\therefore \boxed{F'(z) = e^{-i\theta} (u_r + i v_r)}$$

$$\therefore F'(z) = e^{-i\theta} \left(-\frac{1}{r^2} \cos\theta + i \frac{1}{r^2} \sin\theta \right)$$

$$= -\frac{1}{r^2} e^{-i\theta} (\cos\theta - i \sin\theta) = \frac{1}{r^2} e^{-i\theta} \cdot e^{-i\theta}$$

$$= \frac{1}{r^2} e^{-i2\theta} = \frac{1}{r^2 e^{i2\theta}} = \frac{1}{(r e^{i\theta})^2} = \boxed{\frac{-1}{z^2}}$$

Example for discussions:

check the differentiability by Cauchy-Riemann eqs & derive it; $F(z) = e^z$

-- نهاية محاضرة " Complex numbers, Polar Formulas, Euler eq, Power "

الشكل and Roots, Complex Func, Cauchy-Reiman Eqs

القطبي، معادلو أويلر، قوى وجذور الاعداد المركبة، دالة العدد المركب، معادلات كوشي

-- " ريمن "