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COLLEGE OF ENGINEERING AND TECHNOLOGY
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SUBJECT: ENGINEERING ANALYSIS

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LECTURE ONE

INTRODECTION

In mathematics, the Laplace transform, named after Pierre-Simon Laplace (/ləˈplɑːs/), is an integral transform that converts a function of a real variable (usually, in the time domain) to a function of a complex variable. (in the complex-valued frequency domain, also known as s-domain, or s-plane).

Then also Laplace transform is a mathematical technique that changes a function of time into a function in the frequency domain. If we transform both sides of a differential equation, the resulting equation is often something we can solve with algebraic methods.

LAPLACE TRANSFORMATION

Laplace Transform: **The laplace transform (L.T) is a powerful method for solving differential equations and corresponding initial and boundary value problems. Let f(t) be a time function which is zero for t≤0, and which is defined for t>0. Then, the direct L.T of f(t) denoted** $\mathcal{L}|f(t)| = F(s) = \int_0^\infty f(t).e^{-st}\ dt$ **is defined by:**

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t).e^{-st}\ dt$$

Thus, the operation $\mathcal{L}[\ \ ]$ **transforms f(t), which is in the time domain, into F(s), which is in the complex frequency domain, or simply (S-domain) where S is the complex variable** $(\sigma + j\omega)$**.**

Ex.1: **The laplace transform of the unit step function u(t):**

$$\mathcal{L}[u(t)] = F(s) = \int_0^\infty 1\,.\,e^{-st}\,dt$$

$$= \frac{-1}{s}[e^{-st}]_0^\infty = \int_0^\infty 1\,.\,e^{-st}\,dt$$

$$= \frac{1}{s}$$

Note:

$e^{\infty} = 0$

$e^{0} = 1$

So, $\mathcal{L}[1] = \frac{1}{s}$

Ex.2: **f(t)=3**

$$\mathcal{L}[3] = F(s) = \int_0^\infty 3\,.\,e^{-st}\,dt$$

$$= \frac{-3}{s}[e^{-st}]_0^\infty = \int_0^\infty 1\,.\,e^{-st}\,dt = \frac{3}{s}$$

So, $\mathcal{L}[constant] = \frac{constant}{s}$

Ex.3: f(t)=t

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty t\,.\,e^{-st}\,dt$$

By using part method ($\int udv = uv - \int vdu$),

Let $u = t$ $\qquad dv = e^{-st}\,dt$

$du = dt \qquad v = \frac{1}{-s}\,e^{-st}$

$$= t.\frac{1}{-s}[e^{-st}]_0^\infty - \int_0^\infty \frac{1}{-s}e^{-st}\,dt = \left[\frac{-t}{s}.e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{-t}{s}[e^{-st}]_0^\infty - \left[\frac{1}{s^2}.e^{-st}\right]_0^\infty$$

So, $\mathcal{L}[t] = \frac{1}{s^2}$

Ex.4: $f(t) = t^2$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty t^2\,.\,e^{-st}\,dt$$

Again by using part method ($\int udv = uv - \int vdu$),

Let $u = t^2$ $\qquad dv = e^{-st}\,dt$

$du = 2t\,dt \qquad v = \frac{1}{-s}\,e^{-st}$

$$= t^2.\frac{1}{-s}[e^{-st}]_0^\infty + \int_0^\infty \frac{2}{s}e^{-st}\,.t\,dt$$

The first term is zero, and using the above result of (Ex.3) $\mathcal{L}[t] = \frac{1}{s^2}$ **, the integral reduces to:**

$\mathcal{L}[t^2] = \frac{2}{s^3}$

- In general, if the transforms are worked out for higher power of t, it will found that: $(t) = t^n \implies \mathcal{L}[t] = \frac{n!}{s^{n+1}}$ **,** for s-a>0

Theorem 1: Linearity of the Laplace transformation

For any function f(t) and g(t) whose laplace transform exist and any constant a &b, we have:

$$\mathcal{L}[af(t) \mp bg(t)] = a\mathcal{L}[f(t)] \mp b\mathcal{L}[g(t)]$$

Proof: $\mathcal{L}[af(t) \mp bg(t)]=\int_0^\infty e^{-st}\,[af(t) \mp bg(t)]dt$

$= a\int_0^\infty e^{-st}\, f(t)dt + b\int_0^\infty e^{-st}\, g(t)dt$

$= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

Shift theorem for the Laplace transform

There are two shift theorems, which simplify the task of working with the L.T. The first involves a shift of the variable S to S-a, and the second a shift of the variable t to t-a, where a>0 is arbitrary constant.

Theorem 2: a first shifting theorem

Let $\mathcal{L}[f(t)]$**=F(s), then** $\mathcal{L}[e^{at} f(t)] = F(s-a)$

Proof: $F(s) = \int_0^\infty e^{-st} . f(t)dt$

So, $\mathcal{L}[e^{at} f(t)]$**=** $\int_0^\infty f(t)\, e^{at}\, . e^{-st}\, dt = \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$ **=F(s-a)**

i.e. the multiplication of f(t) by (e^{at}) shifts the variable (s) in the L.T. to (s-a).

Ex.6: let f(t)= cosh at=$\frac{e^{at}+e^{-at}}{2}$. Find $\mathcal{L}[f(t)]$**.**

Sol. From theorem 1,

$\mathcal{L}[cosh\ at] = \frac{1}{2}\mathcal{L}[e^{at}] + \frac{1}{2}\mathcal{L}[e^{-at}] = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right], where\ s > a\ (\geq 0).$

So, $\boxed{\mathcal{L}[cosh\ at]=\frac{s}{s^2-a^2}}$

Ex.7: f(t) = e^{jwt}. $\mathcal{L}[f(t)]$.

Sol. $\mathcal{L}[e^{jwt}] = \int_0^\infty e^{jwt} . e^{-st}\ dt = \int_0^\infty e^{-(s-jw)t}\ dt$

$=\frac{1}{s-jw} x \frac{s+jw}{s+jw} \Longrightarrow \frac{s}{s^2+w^2} + \frac{jw}{s^2+w^2}$

We know that $e^{jwt} = \cos wt + j \sin wt$

$$\mathcal{L}[e^{jwt}] = \mathcal{L}[\cos wt] + j\,\mathcal{L}[\sin wt]$$

Applying linearity theorem,

$\mathcal{L}[cos\ wt] = \frac{s}{s^2+w^2}$ **and** $\mathcal{L}[sin\ wt] = \frac{w}{s^2+w^2}$

Ex.8: f(t)= t . e^{at} . Find $\mathcal{L}[f(t)]$.

$\mathcal{L}(t) = \frac{1}{s^2}$ **, applying the 1st shifting theorem:**

$\mathcal{L}[t.e^{at}] = \frac{1}{(s-a)^2}$

Ex.9: f(t)= $e^{-at}.cost\ wt$

$\mathcal{L}(cos\ wt) = \frac{s}{s^2+w^2}$**, apply 1st shifting theorem:**

$$\mathcal{L}(e^{-at} \cdot cos\ wt) = \frac{s+a}{(s+a)^2+w^2}$$

Exercise: 1) $\mathcal{L}[e^{at}\ cos\ bt] = \frac{s-a}{(s-a)^2+b^2}$

2) $\mathcal{L}[e^{at}\ sin\ bt] = \frac{b}{(s-a)^2+b^2}$

3) $\mathcal{L}[t\ cos\ at] = \frac{s^2-a^2}{(s^2+b^2)^2}$

If $\mathcal{L}[f(t)] = F(s)\ \ and\ g(t) = \begin{cases} f(t-a) \ldots\ldots t > a \\ 0 \ldots..t < a \end{cases}$

i.e $\boxed{\mathcal{L}[g(t)] = e^{-as}\ F(s)}$

Proof: $\mathcal{L}[g(t)] = \int_0^\infty f(t-a)\ e^{-st}\,dt$ **changing the variable in the integral to (t-a)=τ, dt=d τ**

G(s)= $\int_0^\infty f(\tau)\ e^{-s(a+\tau)}\,dt$ = $e^{-as}\int_0^\infty\ e^{-s\tau}.f(\tau)dt = e^{-as}\,F(s)$

Ex.10: The L.T of f(t- $\frac{\pi}{4}$) when f(t)=t sin 2t.

Sol. $\mathcal{L}[f(t)] = \mathcal{L}[t\sin 2t] = \frac{4s}{(s^2+4)^2}$

naw, . $\mathcal{L}\left[f(t-\frac{\pi}{4})\right] = e^{-\frac{\pi}{4s}}\ .\ \frac{4s}{(s^2+4)^2}$

Laplace of [T sin at]
=$\frac{2as}{(s^2+a^2)^2}$

Theorem 4: Change of scale

If F(s) = $\mathcal{L}[f(t)]$

$\therefore\ \mathcal{L}[f(at)]$=$\frac{1}{a}$ **F(**$\frac{s}{a}$**)**

Proof: $\mathcal{L}[f(at)] = \int_0^\infty f(at)\ e^{-st}\,dt$

Let at=u $\Longrightarrow a.dt = du$

$\mathcal{L}[f(at)] = \int_0^\infty f(u)\ e^{-s\frac{u}{a}}\ .\frac{1}{a}$**.du=**$\frac{1}{a}\int_0^\infty f(u)\ e^{-\frac{s\,u}{a}}.du$

=$\frac{1}{a}$ **F(s/a).**

Ex. 11: f(t)=sin 2t

Sol. $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$ & $\mathcal{L}[\sin 2\ t] = \frac{1}{2}\ .\ \frac{1}{\frac{s^2}{4}+1} = \frac{1}{2}.\frac{1}{(s^2+4)/4}$

=$\frac{2}{s^2+4}$

Exercise: f(t)=cos 3t

Table 1 of Laplace transform:

#	F(t)	F(s)
1	1	1/s
2	t	$1/s^2$
3	e^{-at}	$\frac{1}{s+a}$
4	$t\ e^{-at}$	$\frac{1}{(s+a)^2}$
5	Sin wt	$\frac{w}{s^2+w^2}$
6	Cos wt	$\frac{s}{s^2+w^2}$
7	e^{-at} sin wt	$\frac{w}{(s+a)^2+w^2}$ … s>a
8	e^{-at} cos wt	$\frac{s+a}{(s+a)^2+w^2}$ …s>a
9	t sin at	$\frac{2as}{(s^2+a^2)^2}$ $\dots s>0$
10	t cos at	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11	Sinh wt	$\frac{w}{s^2-w^2}$
12	Cosh wt	$\frac{w}{s^2-w^2}$
	Any more…?	….
	….	….

The Inverse Laplace Transform

If $\mathcal{L}[f(t)] = F(s)\ then \implies f(t) = \mathcal{L}^{-1}[F(s)]$, which is called Inverse Laplace Transform (I.L.T). Usually the table of Laplace transform pairs (page 6) are used.

Ex. 13: Given F(s) =$\frac{1}{s-2}$. Find f(t).

Sol. F(t)= $\mathcal{L}^{-1}[F(s)]$=$\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$ (see table 1, #3)

Ex. 14: If F(s) =$\frac{2}{s+2}$. Find f(t).

Sol. F(t)= $\mathcal{L}^{-1}[F(s)]$=$\mathcal{L}^{-1}\left[\frac{2}{s+2}\right] = 2.e^{-2t}$ (see table 1, #3)

Ex. 15: If F(s) =$\frac{10s}{s^2+4} - \frac{3}{s^2+16}$. Find f(t).

Sol. F(s)=10.$\frac{s}{s^2+2^2} - 3\frac{1}{s^2+4^2}$ → $\frac{1}{s^2+4^2} = \frac{1}{4}.\frac{4}{s^2+4^2}$

F(t)= $\mathcal{L}^{-1}[10.\frac{s}{s^2+2^2} - 3.\frac{1}{4}\frac{4}{s^2+4^2}$ = 10 cos 2t - $\frac{3}{4}\sin 4t$ (see table 1, #5&6)

Ex. 16: If F(s) = $\frac{1}{s^2-2s+5}$. Find f(t)

Sol. F(s) =$\frac{1}{s^2-2s+5} = \frac{1}{s^2-2s+1+4}$=$\frac{1}{(s-1)^2+2^2}$

∴F(s)=$\frac{1}{2}$. $\frac{2}{(s-1)^2+2^2}$

$\mathcal{L}^{-1}[\frac{1}{2}.\frac{2}{(s-1)^2+2^2} = \frac{1}{2}e^t \sin 2t$ (See table 1, #7)

Ex. 17: If F(s)= $\frac{4}{s^2}$+$\frac{7}{s^2+1}$. Find f(t), for s>0

Sol. F(t)= $\mathcal{L}^{-1}[F(s)]$=$\mathcal{L}^{-1}\left[\frac{4}{s^2} + \frac{7}{s^2+1}\right]$=4$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 7\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$

$\therefore f(t) = 4t + 7\sin t$ .....(entries 2&5)

Ex. 18: Find f(t) given that F(s)=$\frac{7s^2+18}{s(s^2+9)}$

Sol. Using the partial fractions, re-express F(s) as:

$$\frac{7s^2+18}{s(s^2+9)} = \frac{As}{(s^2+9)} + \frac{B}{s}$$

$$\frac{7s^2+18}{s(s^2+9)} = \frac{As^2+B(s^2+9)}{s(s^2+9)}$$

7s2+18=As2+Bs2+9B

7=(A+B)……………..(1)⟹A=5 and

18=9B……………….(2) ⟹B=18/9=2

Therefore,

F(s)=$\frac{7s^2+18}{s(s^2+9)} = \frac{5s}{(s^2+9)} + \frac{2}{s}$

F(t)= $\mathcal{L}^{-1}[F(s)]$=2$\mathcal{L}^{-1}\frac{1}{s} + 5\mathcal{L}^{-1}\frac{s}{(s^2+9)}$ …. Entries 1&6

$\therefore f(t) = 2 + 5\cos 3t$

Ex. 19: If F(s)=$\frac{s}{s^2+2s+5}$. Find f(t)

Sol. F(s)= $\frac{s}{s^2+2s+5}$=$\frac{s+1-1}{s^2+2s+1+4}$=($\frac{(s+1)-1}{s^2+1)+2^2}$

F(t) =$\mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+2^2} - \frac{1}{2}.\frac{2}{(s+1)^2+2^2}$ entries 7&8

F(t)= e^{-t} cos 2t - $\frac{1}{2}e^{-t}$ sin 2t

Ex. 20: If F(s)= $\frac{s+1}{s^2+s-6}$. Find f(t)

Sol. F(s)= $\frac{s+1}{s^2+s-6} = \frac{s+1}{(s-2)(s+3)}$

$$\frac{s+1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}$$

$$\therefore s + 1 = As + A3 + Bs - 2B$$

$$s + 1 = (A + B)s + 3A - 2B$$

Coefficient of "s"   1=A+B ……………….(1) ∴A=3/5

Coefficient of "s^0"   1=3A-2B ……………….(2) ∴B=2/5

$$\therefore F(s) = \frac{1}{5}\left\{\frac{3}{s-2} + \frac{2}{s+3}\right\}$$

$\therefore f(t) = \frac{1}{5}\mathcal{L}^{-1}[\frac{3}{s-2}+\frac{2}{s+3}] \Rightarrow f(t) = \frac{1}{5}(3e^{2t} + 2e^{-3t})$

Laplace Transformations of Derivatives and integrals

Suppose that f(t) is continuous for all t≥ 0, and has derivative f' (t) which is continues on every finite interval in the range t≥ 0. Then, the L.T. of the derivative f' (t) exists:

If F(s) = $\mathcal{L}[f(t)]$

Then $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Proof: $\mathcal{L}[f'(t)]=\int_0^\infty f'(t).\ e^{-st}\ dt$

Integrating by parts ($\int udv = uv - \int vdu$)

dv=f' (t) dt   & $u=e^{-st}$

v=f(t)   &   du= -s . e^{-st} dt

$\mathcal{L}[f'(t)] = [e^{-st}\ [f(t)]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$

$$\therefore \mathcal{L}[f'(t)] = sF(s) - f(0)$$

For high order derivative:

$$\mathcal{L}[f''(t)] = s^2F(s) - s\,F(0) - f'(0)$$

And,

$$\mathcal{L}[f'''(t)] = s^3F(s) - s^2\,F(0) - sf'(0) - f''(0)$$

Ex.21: **F(t)=(sin t)^2 . Find F(s), if f(0)=0**

Sol. F'(t)=2 sin t. cos t= sin 2t

f'(t)=sF(s) –f(0)

$$\mathcal{L}[f'(t)] = \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$$

$$\frac{2}{s^2+4} = sF(s) - 0 \quad or \ \Rightarrow \qquad F(s) = \frac{2}{s(s^2+4)} = \mathcal{L}(\sin t)^2$$

Ex.22: let f(t)=t . sin wt. Fond F(s), where f(0)=0

Sol. F ' (t)=sin wt .1+w.t.cos wt$\Rightarrow f'(0) = 0$

f ''(t) =coswt. W+[wt(-sin wt. w)+cos wt .w]

f ''(t)=2w cos wt-w^2 t sin wt

$$\therefore f''(t) = 2\,w\cos wt - w^2 f(t)$$

$$\mathcal{L}[f''(t)] = s^2F(s) - sF(0) - f\,'(0)$$

$$\mathcal{L}[f''(t)] = 2w\,\mathcal{L}[\cos wt] - w^2\mathcal{L}[\mathrm{f(t)}]$$

$$\therefore 2w\mathcal{L}[\cos wt] - w^2\mathcal{L}[\mathrm{f(t)}] = \mathrm{s}^2F(s) - 0 - 0$$

$$2\mathrm{w}.\frac{s}{s^2+w^2} - w^2F(s) = s^2F(s)$$

$$\frac{2ws}{s^2+w^2} = s^2F(s) + w^2F(s) \Rightarrow F(s) = \frac{2ws}{(s^2+w^2)^2}$$

Ex.23: Find $\mathcal{L}$[Y''], given that Y(t)=sin 2t

Sol. Y(t)=sin 2t $\Rightarrow$y(0)=0

Y '(t)=2 cos 2t $\Rightarrow$y '(0)=2

Y''(t)= -4 sin 2t $\Rightarrow$y '' (0)=0

Thus,

$$\mathcal{L}[\mathrm{Y}'''] = s^3\,\mathcal{L}[y(t)] - s^2y(0) - sy'(0) - y''(0)$$

$$s^3\,\mathcal{L}[\sin 2t] - 0\mathcal{L} - 2s - 0$$

By using table 1 (#5),

$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$, for s>0

So that,

$$\mathcal{L}[Y'''] = \frac{2s^3}{s^2+4} - 2s = \frac{2s^3 - 2s(s^2+4)}{s^2+4} = \frac{-8s}{s^2+4}$$

$$\therefore\ \mathcal{L}[Y'''] = -8\cos 2t\ (using\ table\ 1\ \#6)$$

Ex.24: solve Y''-3 Y' +2Y=e^{-t} using L.T. Given that y(0)=1, y '(0)=0

Sol. Take the L.T. of both sides of the equation to obtain:

$\mathcal{L}[Y''] - 3\mathcal{L}[Y'] + 2\mathcal{L}[Y] = \mathcal{L}[e^{-t}]$

Transforming the derivatives, and transforming e^{-t} by means of entry "3" of table 1, we obtain:

$[s^2Y(s) - y'(0) - sy(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$

$[s^2Y(s) - 0 - s] - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s+1}$

$$\therefore Y(s)[s^2 - 3s + 2] - s + 3 = \frac{1}{s+1}$$

$$Y(s)[s^2 - 3s + 2] = Y(s)\frac{1}{s+1} + s - 3 = \frac{1 + s(s+1) - 3(s+1)}{s+1}$$

$$= \frac{s^2 - 2s - 2}{s+1}$$

$$\therefore Y(s) = \frac{s^2 - 2s - 2}{(s+1)(s^2 - 3s + 2)} = \frac{s^2 - 2s - 2}{(s+1)(s-2)(s-1)}$$

Using partial fractions,

$Y(s)=\frac{3}{2}\left(\frac{1}{s-1}\right)-\frac{2}{3}\left(\frac{1}{s-2}\right)+\frac{1}{6}(\frac{1}{s+1})$

Taking the inverse L.T using entry "3" of table 1:

$Y(t)=\frac{3}{2}e^{t}-\frac{2}{3}e^{2t}+\frac{1}{6}e^{-t}$

$$Hint: \frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{c}{s+1}$$

$$=\frac{A[(s-2)(s+1)+B[(s-1)(s+1)]+c[(s-2)(s-1)]}{(s-1)(s-2)(s-3)}$$

$$=\frac{A[s^2+s-2s-2]+B[s^2+s-s-1]+c[s^2-s-2s+2]}{(s-1)(s-2)(s-3)}=\frac{A[s^2-s-2]+B[s^2-1]+c[s^2-3s+2]}{(s-1)(s-2)(s-3)}$$

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}=\frac{As^2+As-2A+Bs^2-B+cs^2-3cs+2c]}{(s-1)(s-2)(s-3)}$$

Coeff. of s^2 1=A+B+C......................(1)$\Longrightarrow A=3/2$

Coeff. of s -2=-A-3C.........................(2)$\Longrightarrow B=-2/3$

Coeff. of s^0 -2=-2A-B+2c.................(3)$\Longrightarrow C=1/6$

$$Y(s)=\frac{\frac{3}{2}}{(s-1)}-\frac{\frac{2}{3}}{(s-2)}+\frac{\frac{1}{6}}{(s-1)}$$

Laplace transform of integration

If F(s)= $\mathcal{L}$[f(t)]

then, $\mathcal{L}\left[\int_0^{\infty} f(t)\ dt\right]=\int_0^{\infty}[f(t)\,dt].\ e^{-st}\ dt$

let u= $f(t)\,dt$ $\qquad dv=e^{-st}dt$

du=f(t) dt $\qquad v=-\frac{1}{s}e^{-st}$

$$\mathcal{L}\left[\int_0^t f(t)\ dt\right]=\{[\int_0^t f(t)dt]\left[-\frac{1}{s}e^{-st}\right]\}_0^{\infty}-\int-\frac{1}{s}e^{-st}f(t)dt$$

$$=-\frac{1}{s}e^{-st}\int_0^t f(t)\ dt]_0^\infty + \frac{1}{s}.F(s)$$

Since $e^{-st} \Longrightarrow 0\ \ as\ t \Longrightarrow \infty$

& t$\Longrightarrow$ 0, the integral in this term vanishes

$$\therefore \mathcal{L}\left[\int_0^t f(t)\ dt\right] = \frac{F(s)}{s}$$

Multiplication by power of t

If F(s)= $\mathcal{L}$[f(t)],

$\therefore \mathcal{L}[t^n\, f(t)] = (-1)^n \frac{d^n}{ds^n}\, F(s)$, where n=1, 2, 3,...

Proof:

For n=1, F(s)= $\mathcal{L}[\int_0^\infty f(t)\ dt]$

, $\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty f(t)\ e^{-st}dt = \int_0^\infty \frac{d}{ds}\ e^{-st}f(t)dt$

$=\int_0^\infty -t\ e^{-st}f(t)dt = \int_0^\infty -e^{-st}[t\, f(t)]dt = -\ \mathcal{L}[\text{t f(t)}]$

Or $\mathcal{L}[\text{ t f(t)}] = -\frac{dF}{ds} = -F'(s)$

Ex. 25: f(t)=t cos 3t. Find $\mathcal{L}$[f(t)]

Proof: $\mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$

$$\mathcal{L}[\text{t} \cos 3t] = (-1)\ \frac{d}{ds} . \frac{s}{s^2+9}$$

$=\frac{(s^2+9).1-s.2s}{(s^2+9)^2}.(-1) = \frac{s^2-9}{(s^2+9)^2}$

Ex.26: Given $\int_0^t t\cos 2t \quad dt. find \quad \mathcal{L}$ of the function

H.W

Sol. $\mathcal{L}[\text{t}\cos 2t] = \frac{s^2-4}{(s^2+4)^2}$

$$\therefore \mathcal{L}[\int_0^t t\cos 2t] = \frac{1}{s}.F(s) = \frac{1}{s}.\frac{s^2-4}{(s^2+4)^2}$$

Ex.27: Given F(s)=$\frac{1}{s^2(s^2+w^2)}$. Find f(t)

Sol. We have $\mathcal{L}^{-1}[\frac{1}{s}.\left(\frac{1}{s^2+w^2}\right) = \frac{1}{w}\int_0^t \sin wt\ dt = \frac{1}{w}(1-\cos wt)$

Applying the integration theorem once more, we obtain the desired answer:

$$\mathcal{L}^{-1}[\frac{1}{s^2}\left(\frac{1}{s^2+w^2}\right) = \frac{1}{w}\int_0^t (1-\cos wt)\,dt = \frac{1}{w^2}(t-\frac{sinwt}{w})$$

Note: you can also resolve it by partial fraction method ...etc