ALMUSTAQBAL UNIVERSITY

COLLEGE OF ENGINEERING AND TECHNOLOGY

COMPUTER ENGINEERING TECHNIQUE DEPARTMENT

SUBJECT: ENGINEERING ANALYSIS

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LECTURE THREE

APPLICATION OF LAPLACE TRANSFORM

SOLUTION OF DIFFERENTIAL EQUATION BY USING LAPLACE TRANSFORM

Theorem :

If $f(t)$ is continuous in $t \geq o$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq o$ and $f(t)$ and $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof :

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$.

By definition, $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$

$$= \int_0^\infty e^{-st} d(f(t))$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt \text{, on integration by parts}$$

$$= \lim_{t \to \infty}\left[e^{-st} f(t)\right] - f(0) + s.L(f(t))$$

$$= 0 - f(0) + sL(f(t)) \qquad [\because f(t) \text{ is of exponential order}]$$

$$= sL(f(t)) - f(0)$$

$$L(f'(t)) = sL(f(t)) - f(0)$$

In the above theorem if we replace $f(t)$ by $f'(t)$ we get,

$$L(f''(t)) = sL(f'(t)) - f'(0)$$

$$= s\left[sL(f(t)) - f(0)\right] - f'(0)$$

$$= s^2 L(f(t)) - sf(0) - f'(0)$$

Repeated application of the above theorem gives the following result:

$$L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f^1(0) - \ldots - f^{n-1}(0)$$

Solved Problems :

1. Using Laplace transform, solve $y' - y = t, y(0) = 0$.

Solution :

Given $y' - y = t, y(0) = 0$

Taking Laplace transfrom on both sides,

$$L(y') - L(y) = L(t)$$

$$sL(y) - y(0) - L(y) = \frac{1}{s^2}$$

$$L(y)\left[S - 1\right] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1}\right) dt\ dt$$

$$y = \int_0^t \int_0^t e^t\ dt\ dt$$

$$= \int_0^t \left[e^t\right]_0^t dt$$

$$= \int_0^t \left[e^t - 1\right]_0^t dt$$

$$= \left(e^t - t\right)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$.

Solution :

Taking Laplace transforms on the sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$\left[s^2 L(y) - sy(0) - y^1(0)\right] - 4\left[sL(y) - y(0)\right] + 8L(y) = \frac{1}{s-2}$$

$$i.e., \left[s^2 - 4s + 8\right]L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

Solving we get $A=\frac{1}{4}, \quad B=\frac{-1}{4}, \quad C=\frac{1}{2}$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{-1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{S-2}+\frac{\frac{7}{4}(S-2)-6}{(S-2)^2+4}$$
$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}\left(\frac{\frac{7}{4}s-6}{s^2+4}\right)$$
$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$
$$=\frac{1}{4}e^{2t}\left(1+7\cos 2t-12\sin 2t\right)$$

3. Use Laplace transform to solve $y'-y=e^t$ given that $y(0)=1$

Solution:

$$y'-y=e^t$$

Taking Laplace transform on both sides of the equation, we get $y'-y=t, y(0)=0$

$$\left[sL(y)-y(0)\right]-L(y)=\frac{1}{s-1}$$
$$L(y)[s-1]=\frac{1}{s-1}+1$$
$$L(y)=\frac{s}{(s-1)^2}$$
$$y=L^{-1}\left[\frac{s}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{1}{(s-1)^2}\right]$$
$$=e^t+te^t$$
$$=e^t(1+t)$$

4. Solve $\frac{d^2y}{dt^2}+9y=18t$ given that $y(0)=0=y\left(\frac{\pi}{2}\right)$

Solution :

$$y''+9y=18t \qquad \text{where} \quad y''=\frac{d^2y}{dt^2}$$

Taking Laplace transform on both sides of the equation, we get

$$L(y'')+9L(y)=18L(t)$$

$$\left[s^2L(y)-sy(0)-y'(0)\right]+9L(y)=\frac{18}{s^2}$$

$$L(y)\left[s^2+9\right]=\frac{18}{s^2}+y'(0) \qquad [\because y'(0) \text{ is not given we can take it to be a constant a}]$$

$$=\frac{18}{s^2}+a$$

$$=\frac{as^2+18}{s^2}$$

$$L(y)=\frac{as^2+18}{s^2(s^2+9)}$$

$$=\frac{a}{s^2+9}+\frac{18}{s^2(s^2+9)}$$

$$y=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{18}{s^2(s^2+9)}\right)$$

$$=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{2}{s^2}-\frac{2}{(s^2+9)}\right) \qquad \text{(using partial fractions)}$$

$$=\frac{a\sin 3t}{a}+2t-\frac{2\sin 3t}{3}$$

Now, using the conditions $y=0$ and $t=\frac{\pi}{2}$ we have

$$0=\frac{a}{3}\sin\left(\frac{3\pi}{2}\right)+\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$=-\frac{a}{3}+\pi+\frac{2}{3}$$

$$\frac{a}{3}=\frac{3\pi+2}{3}$$

Hence $a=3\pi+2$

$$\therefore y=\frac{(3\pi+2)\sin 3t}{3}+2t-\frac{2\sin 3t}{3}$$

$$=\pi\sin 3t+2t$$

Exercise :

1. Solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$ and $y'(0) = -2$

2. Solve $y'' + 4y = \sin wt$, $y(0) = 0$ and $y'(0) = 0$

3. Solve $y'' + y' - 2y = 3\cos 3t - 11\sin 3t$, $y(0) = 0$ and $y'(0) = 6$

4. Solve $(D^2 + 4D + 13)y = e^{-t}\sin t$, $y = 0$ and $Dy = 0$ at $t = 0$ where $D = \frac{d}{dt}$

5. Solve $(D^2 + 6D + 9)x = 6t^2 e^{-3t}$, $x = 0$ and $Dx = 0$ at $t = 0$.

6. Solve $x'' + 3x' + 2x = 2(t^2 + t + 1)$, $x(0) = 2$, $x'(0) = 0$.

7. Solve $y'' - 3y' - 4y = 2e^t$, $y(0) = y'(0) = 1$.

8. Solve $x'' + 9x = 18t$, $x(0) = 0$, $x\left(\frac{\pi}{2}\right) = 0$

9. $y'' + 4y' = \cos 2t$, $y(\pi) = 0$, $y'(\pi) = 0$.

10. $x'' - 2x' + x = t^2 e^{-3t}$, $x(0) = 2$, $x'(0) = 3$.

Answers :

1. $y = \frac{1}{4}e^{2t}(1 + 7\cos 2t - 12\sin 2t)$
2. $y = \frac{1}{8}(\sin 2t - 2t\cos 2t)$
3. $y = \sin 3t - e^{-2t} + e^t$
4. $y = \frac{1}{85}\left[e^{-t}\{-2\cos t + 9\sin t\}\right] + e^{-2t}\left\{2\cos 3t - \frac{7}{3}\sin 3t\right\}$
5. $x = \frac{1}{2}t^4 e^{-3t}$
6. $x = t^2 - 2t + 3 - e^{-2t}$
7. $y = \frac{1}{25}(13e^{-t} - 10te^{-t} + 12e^{4t})$
8. $x = 2t + \pi\sin 3t$
9. $y = \frac{1}{4}(t - \pi)\sin 2t$
10. $x = \left(\frac{t^4}{12} + t + 2\right)e^t$

Solution of Integral equations using Laplace transform

Theorem :

If $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Proof :

Let $g(t) = \int_0^t f(t)dt$

$$\therefore \quad g^1(t) = f(t)$$

$$\therefore \quad L(g^1(t)) = sL(g(t)) - g(0)$$

$$i.e \quad L(f(t)) = sL\left(\int_0^t f(t)dt\right) - \int_0^0 f(t)dt$$

$$\therefore \quad L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Corollary :

$$L\left[\int_0^t \int_0^t f(t)dtdt\right] = \frac{1}{s^2}L(f(t))$$

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$$L\left[\int_0^t \int_0^t .....\int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L(f(t))$$

Problems :

1. Solve $y + \int_0^t ydt = t^2 + 2t$

Solution :

Given $y + \int_0^t ydt = t^2 + 2t$

Taking Laplace Transform on both sides

$$L(y) + L\left(\int_0^t ydt\right) = L(t^2) + L(2t)$$

$$L(y) + \frac{1}{s}L(y) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L(y)\left[1 + \frac{1}{s}\right] = 2\left[\frac{1+s}{s^3}\right]$$

$$L(y)\left[\frac{s+1}{s}\right]=2\left[\frac{s+1}{s^3}\right]$$

$$L(y)=2\left[\frac{s+1}{s^3}\right]\left[\frac{s}{s+1}\right]$$

$$=\frac{2}{s^2}$$

$$y=L^{-1}\left(\frac{2}{s^2}\right)=2t$$

2. Solve $\frac{dy}{dt}+2y+\int_0^t ydt=2\cos t, \quad y(0)=1$

Solution :

Given $y'+2y+\int_0^t ydt=2\cos t$

Taking Laplace Transform on both sides

$$L(y')+2L(y)+L\left(\int_0^t ydt\right)=2L(\cos t)$$

$$sL(y)-y(0)+2L(y)+\frac{1}{s}L(y)=\frac{2s}{s^2+1}$$

$$L(y)\left[s+2+\frac{1}{s}\right]-1=\frac{2s}{s^2+1}$$

$$L(y)\left[\frac{s^2+2s+1}{s}\right]=\frac{2s}{s^2+1}+1$$

$$L(y)=\left[\frac{s^2+2s+1}{s^2+1}\right]\left[\frac{s}{s^2+2s+1}\right]$$

$$=\frac{s}{s^2+1}$$

$$y=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

3. Using Laplace Transform solve $y+\int_0^t y(t)dt=e^{-t}$

Solution :

Given $y+\int_0^t y(t)dt=e^{-t}$

Taking Laplace transform on both sides,

$$L(y)+L\left(\int_0^t y(t)dt\right)=L(e^{-t})$$

$$L(y)+\frac{1}{s}L(y)=\frac{1}{s+1}$$

$$L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y=L^{-1}\left(\frac{s}{(s+1)^2}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$=L^{-1}\left(\frac{1}{s+1}\right)-e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y=e^{-t}-e^{-t}t$$

$$y=e^{-t}(1-t)$$

4. Using Laplace transform, solve $x+\int_0^t x(t)dt=\cos t+\sin t$

Solution :

$$x+\int_0^t x(t)dt=\cos t+\sin t$$

Taking Laplace transform on both sides,

$$L(x)+L\left(\int_0^t x(t)dt\right)=L(\cos t+\sin t)$$

$$L(x)\left[1+\frac{1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)\left[\frac{s+1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)=\left(\frac{s+1}{s^2+1}\right)\left(\frac{s}{s+1}\right)$$

$$L(x)=\frac{s}{s^2+1}$$

$$\therefore x=L^{-1}\left(\frac{s}{s^2+1}\right)=\cos t$$

Solving Integral Equations using convolution

Theorem :

By the definition of convolution, we have $f(t)*g(t)=\int_0^t f(u)g(t-u)du$

and by convolution theorem, $L(f(t)*g(t))=L(f(t))L(g(t))$

Problems :

1. Solve $y=1+2\int_0^t e^{-2u}y(t-u)\ du$ ______(1)

Solution :

$\int_0^t e^{-2u}y(t-u)\ du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t)=e^{-2t},\ \ g(t)=y(t)$

Taking Laplace Transform on both sides of (1),

$$L(y)=L(1)+2L\left[\int_0^t e^{-2u}y(t-u)\ du\right]$$

$$=\frac{1}{s}+2L\left[e^{-2t}*y(t)\right] \qquad \text{(Definition of convolution)}$$

$$=\frac{1}{s}+2L(e^{-2t})L(y) \qquad \text{(Convolution theorem)}$$

$$=\frac{1}{s}+2\left(\frac{1}{s+2}\right)L(y)$$

$$L(y)=\frac{1}{s}+\frac{2}{s+2}L(y)$$

$$L(y)\left[1-\frac{2}{s+2}\right]=\frac{1}{s}$$

$$L(y)\left[\frac{s}{s+2}\right]=\frac{1}{s}$$

$$L(y)=\frac{s+2}{s^2}=\frac{1}{s}+\frac{2}{s^2}$$

$$y=L^{-1}\left(\frac{1}{s}+\frac{2}{s^2}\right)$$

$$y=1+2t$$

2. Using Laplace transform solve $y=1+\int_0^t y(u)\sin(t-u)du$

Solution :

Given $y=1+\int_0^t y(u)\sin(t-u)du$

Taking Laplace transform on both sides,

$$L(y) = L(1) + L\left[\int_0^t y(u)\sin(t-u)du\right] \quad \_\_\_\_(1)$$

Now the integral $\int_0^t y(u)\sin(t-u)du$ is of the form $\int_0^t f(u)\,\mathrm{g}(t-u)du$ where $f(t) = y(t), \quad g(t) = \sin t$

$\therefore$ (1) becomes

$$L(y) = \frac{1}{s} + L(y(t) * \sin t)$$

$$L(y) = \frac{1}{s} + L(y)\cdot\frac{1}{s^2+1}$$

$$L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s}$$

$$L(y) = \frac{s^2+1}{s^3}$$

$$= \frac{1}{s} + \frac{1}{s^3}$$

$$y = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^3}\right)$$

$$y = 1 + \frac{1}{2}t^2$$

3. Using Laplace transform, solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Solution :

Given that $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$ $\_\_\_\_\_(1)$

Taking Laplace transform on both sides of (1),

$$L(f(t)) = L(\cos t) + L\left[\int_0^t e^{-u} f(t-u)du\right]$$

$$= \frac{s}{s^2+1} + L(e^{-t} * f(t))$$

$$= \frac{s}{s^2+1} + L(e^{-t})L(f(t))$$

$$= \frac{s}{s^2+1} + \frac{1}{s+1}L(f(t))$$

$$L(f(t))\left[1-\frac{1}{s+1}\right]=\frac{s}{s^2+1}$$

$$L(f(t))\left[\frac{s}{s+1}\right]=\frac{s}{s^2+1}$$

$$L(f(t))=\frac{s+1}{s^2+1}$$

$$f(t)=L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$f(t)=\cos t+\sin t$$

4. Solve the integral equation $y(t)=t^2+\int_0^t y(t)\sin(t-u)du$

Solution :

$$y(t)=t^2+\int_0^t y(t)\sin(t-u)du$$

Taking Laplace transform obn both sides,

$$L(y(t))=L(t^2)+L\left[\int_0^t y(t)\sin(t-u)du\right]$$

$$L(y)=\frac{2}{s^3}+L(y(t)*\sin t)$$

$$=\frac{2}{s^3}+L(y)L(\sin t)$$

$$=\frac{2}{s^3}+L(y)\left(\frac{1}{s^2+1}\right)$$

$$L(y)\left(1-\frac{1}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)\left(\frac{s^2}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)=\frac{2(s^2+1)}{s^5}=\frac{2}{s^3}+\frac{2}{s^5}$$

$$y=L^{-1}\left(\frac{2}{s^3}\right)+\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)$$

$$y=t^2+\frac{1}{12}t^4$$

Exercise :

1. Solve $x' + 3x + 2\int_0^t x\,dt = t, \quad x(0) = 0$

2. Solve $y' + 4y + 5\int_0^t y\,dt = e^{-t} \quad y(0) = 0$

3. Solve $x' + 2x + \int_0^t x\,dt = \cos t, \quad x(0) = 1$

4. Solve $y' + 4y + 13\int_0^t y\,dt = 3e^{-2t}\sin 3t, \quad y(0) = 3$

5. Solve $x(t) = 4t - 3\int_0^t x(u)\sin(t-u)du$

6. Solve $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)du$

7. Solve $\int_0^t y(u)y(t-u)du = 2y(t) + t - 2$

8. Solve $y(t) = t + \int_0^t \sin u\; y(t-u)du$

9. Solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

10. Solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Answers :

1. $x = \frac{1}{2}(1+e^{-2t}) - e^{-t}$

2. $y = \frac{-1}{2}e^{-t} + \frac{1}{2}e^{-t}(\cos t + 3\sin t)$

3. $x = \frac{1}{2}\left[(1-t)e^{-t} + \cos t\right]$

4. $y = e^{2t}\left[3\cos 3t - \frac{7}{3}\sin 3t + \frac{3}{2}t\sin 3t + t\cos 3t\right]$

5. $x = t + \frac{3}{2}\sin 2t$

6. $y(t) = e^{-t}\left(1-t\right)^2$

7. $y(t) = 1$

8. $y = t + \frac{t^3}{6}$

9. $y = 1 + \frac{t^2}{2}$

10. $f(t) = \cos t + \sin t$

Simulaneous differential equations

1. Using Laplace transform solve

$$\frac{dx}{dt} + y = \sin t$$
$$\frac{dy}{dt} + x = \cos t$$

given $x(0) = 2$ and $y(0) = 0$

Solution :

Applying Laplace transform to the given equations

We get, $L(x') + L(y) = L(\sin t)$

$L(y') + L(x) = L(\cos t)$

$$\therefore sL(x) - x(0) + L(y) = \frac{1}{s^2+1}$$
$$sL(y) - y(0) + L(x) = \frac{s}{s^2+1}$$
$$\therefore sL(x) + L(\mathrm{y}) = \frac{1}{s^2+1} + 2$$
$$= \frac{2s^2+3}{s^2+1} \quad \text{\_\_\_\_(1)}$$

Also $sL(y) + L(x) = \dfrac{s}{s^2+1}$ _____(2)

$$(1) \times s \Rightarrow s^2 l(x) + sL(y) = \frac{(2s^2+3)s}{s^2+1} \quad \text{\_\_\_\_\_(3)}$$
$$(2) \Rightarrow \quad L(x) + sL(y) = \frac{s}{s^2+1} \quad \text{\_\_\_\_\_(4)}$$
$$(3)\text{ - }(4) \quad (s^2-1)L(x) = \frac{(2s^2+3)}{s^2+1} - \frac{s}{s^2+1}$$
$$= \frac{2s^3+2s}{s^2+1}$$
$$L(x) = \frac{2s}{s^2-1} \quad \text{\_\_\_\_\_(5)}$$

Substituting (5) in (2), we get

$$sL(y) = \frac{s}{s^2+1} - \frac{2s}{s^2-1} = \frac{s(s^2-1) - 2s(s^2+1)}{(s^2+1)(s^2-1)}$$
$$= \frac{-s^3-3s}{(s^2+1)(s^2-1)}$$
$$= \frac{-s(s^2+3)}{-(s^2+1)(1-s^2)}$$
$$L(y) = \frac{(s^2+3)}{(s^2+1)(1-s^2)} \quad \text{\_\_\_\_\_(6)}$$

From (5),
$$x = L^{-1}\left(\frac{2s}{s^2-1}\right)$$
$$= 2\cosh t$$
$$y = L^{-1}\left(\frac{(s^2+3)}{(1-s^2)(s^2+1)}\right)$$

Consider $\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{A}{1-s} + \frac{B}{1+s} + \frac{Cs+D}{s^2+1}$ ____(7)

$$s^2+3 = A(1+s)(s^2+1) + B(1-s)(s^2+1) + (Cs+D)(1-s)(1+s)$$

Put $s=1,\quad 4 = A(2)(2)$

$$\Rightarrow 4 = 4A \Rightarrow A = 1$$

Put $s=-1,\quad 4 = B(2)(2)$

$$\Rightarrow B = 1$$

Put $s=0,\quad 3 = A+B+D$

$$3 = 1+1+D$$
$$\Rightarrow D = 1$$

Comparing the coefficient of S,

$$0 = A - B + C$$
$$\Rightarrow C = 0$$

Substituting the values of A, B, C, D in (7) we get

$$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{1}{1-s} + \frac{1}{1+s} + \frac{1}{s^2+1}$$
$$\therefore y = L^{-1}\left(\frac{1}{1-s}\right) + L^{-1}\left(\frac{1}{1+s}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$y = -e^{-t} + e^{-t} + \sin t$$

Hence the solution is $x = 2\cos ht$ and $y = -e^{-t} + e^{-t} + \sin t$

2. Solve $\frac{dx}{dt} + ax = y$

$$\frac{dy}{dt} + ay = x$$

given that $x = 0$ and $y = 1$ when $t = 0$

Solution :

Applying Laplace transform we get

$$L(x') + aL(x) = L(y)$$
$$L(y') + aL(y) = L(x)$$
$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$

Given that $x(0) = 0, \quad y(0) = 1$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$
$$\therefore sL(x) + aL(x) = L(y)$$
$$sL(y) - 1 + aL(y) = L(x)$$
$$\therefore (s+a)L(x) = L(y)$$
$$(s+a)L(x) - L(y) = 0 \qquad \_\_\_\_(1)$$
$$-L(x) + (s+a)L(y) = 1 \qquad \_\_\_\_(2)$$
$$(1) + (s+a) \times (2) \Rightarrow L(y)\left[(s+a)^2 - 1\right] = s + a$$
$$\therefore L(y) = \frac{s+a}{(s+a)^2 - 1}$$

Also by (1) $L(x) = \dfrac{1}{(s+a)^2 - 1}$

$$\therefore x = L^{-1}\left(\frac{1}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{1}{s^2 - 1}\right)$$
$$= e^{-at} \sin ht$$
$$y = L^{-1}\left(\frac{s+a}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{s}{s^2 - 1}\right)$$
$$= e^{-at} \cos ht$$