Resultant It is the result of adding **two** or more **vectors** together.

Example: Find the **Resultant** for the vector force where $\overrightarrow{F_1} = 100i + 150j + 200k \ and \ \overrightarrow{F_2} = 50i - 30j + 50k$

$$\overrightarrow{F_1} + \overrightarrow{F_2} = (100i + 150j + 200k) + (\overrightarrow{F_2} = 50i - 30j + 50k) \quad \Rightarrow \overrightarrow{F_1} + \overrightarrow{F_2} = 150\,i + 120j + 250k$$

Dot Product: it is symbolized as $(dot \overset{or}{\Longrightarrow} \bullet)$

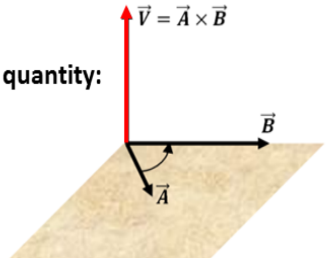
The dot product of any tow vectors $\vec{A} = x_1 i + y_1 j + z_1 k$ and $\vec{B} = x_2 i + y_2 j + z_2 k$ is scalar quantity:

$$\vec{A}(dot)\vec{B} \overset{or}{\Longrightarrow} \vec{A}.\vec{B} \overset{Scalar}{\Longrightarrow} C = x_1x_2 + y_1y_2 + z_1z_2$$

Example: $if\ \vec{A} = 2i + 7j + 4k \ \ and\ \vec{B} = 3i - j + 5k$ find $A(dot)\,B$:

$\vec{A} = 2i + 7j + 4k$
$\vec{B} = 3i - j + 5k$ $\quad A(dot)\,B \overset{or}{\Longrightarrow} A.B = (2)(3) + (7)(-1) + (4)(5) \quad \Rightarrow A.B = 6 - 7 + 20 \Rightarrow A.B = 19$

Cross Product: it is symbolized as $(cross \overset{or}{\Longrightarrow} \times)$



The cross product of any two vectors $\vec{A} = x_1 i + y_1 j + z_1 k$ and $\vec{B} = x_2 i + y_2 j + z_2 k$ is vector quantity:

$$\vec{A} cross\, \vec{B} \overset{or}{\Longrightarrow} \vec{A} \times \vec{B} \qquad \overset{Vactor}{\Longrightarrow} \vec{V} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\Rightarrow \vec{V} = (y_1z_2 - y_2z_1)i - (x_1z_2 - x_2z_1)j + (x_1y_2 - x_2y_1)k \quad \overset{This\ Result\ that\ Vactor}{\Longrightarrow} \vec{V} \perp \vec{A}\ \&\ \vec{V} \perp \vec{B}$$

Example: $if\ \vec{A} = 2i + j + 4k \ \ and\ \vec{B} = 3i - j + 5k$ find $\vec{A} \times \vec{B}\ and\ \vec{B} \times \vec{A}$:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 4 \\ 3 & -1 & 5 \end{vmatrix} \Rightarrow \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 4 \\ 3 & -1 & 5 \end{vmatrix} \begin{matrix} i & j \\ 2 & 1 \\ 3 & -1 \end{matrix} \Rightarrow \vec{A} \times \vec{B} = (5i + 12j - 2k) - (3k - 4i + 10j)$$

$$\Rightarrow \vec{A} \times \vec{B} = 9i + 2j - 5k$$

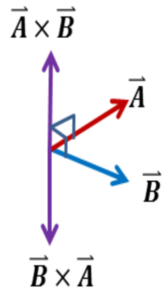
$$\vec{B} \times \vec{A} = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 2 & 1 & 4 \end{vmatrix} \Rightarrow \vec{B} \times \vec{A} = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} i & j \\ 3 & -1 \\ 2 & 1 \end{matrix} \Rightarrow \vec{B} \times \vec{A} = (-4i + 10j + 3k) - (-2k + 5i + 12j)$$

$$\Rightarrow \vec{B} \times \vec{A} = -9i - 2j + 5k$$

$$\Rightarrow \vec{A} \times \vec{B} = 9i + 2j - 5k$$

$$\Rightarrow \vec{B} \times \vec{A} = -9i - 2j + 5k$$

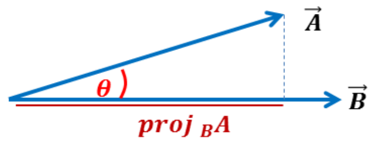
Note that $\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$



Angle between two vectors: The angle between any two vectors such as $\vec{A}\ and\ \vec{B}$ can be finding by using the following formulas:

$$\cos\theta = \frac{\vec{A}\ .\ \vec{B}}{|\vec{A}|\ |\vec{B}|} \Rightarrow \theta = \cos^{-1}\frac{\vec{A}\ .\ \vec{B}}{|\vec{A}|\ |\vec{B}|} \quad or\ by \quad \sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}|\ |\vec{B}|} \Rightarrow \theta = \sin^{-1}\frac{|\vec{A}\times\vec{B}|}{|\vec{A}|\ |\vec{B}|}$$

Projection: the vector projection of $\vec{A}\ onto\ \vec{B}$ is: $proj_{\,B}A = \left(\frac{\vec{A}\ .\ \vec{B}}{|B|}\right)\frac{\vec{B}}{|B|}$ *is vector projection of* $\vec{A}$ *onto* $\vec{B}$



$proj_{\,B}A = \frac{\vec{A}\ .\ \vec{B}}{|B|}$ *is scalar component of* $\vec{A}$ *in the direction of* $\vec{B}$

Example: if $\vec{A} = 3i - j + 2k\ and\ \vec{B} = 2i - 2j + k$ find The angle between $\vec{A}\ and\ \vec{B}$ and projection of $\vec{A}\ onto\ \vec{B}$

$$\theta = \cos^{-1}\frac{\vec{A}\ .\ \vec{B}}{|\vec{A}|\ |\vec{B}|} \Rightarrow \cos^{-1}\frac{6+2+2}{\sqrt{9+1+4}\ \sqrt{4+4+1}} \Rightarrow \theta = \cos^{-1}\frac{10}{\sqrt{126}} \Rightarrow \theta = 27.02^{o}$$

$$proj_{\,B}A = \left(\frac{\vec{A}\ .\ \vec{B}}{|B|}\right)\frac{\vec{B}}{|B|} \Rightarrow \left(\frac{6+2+2}{\sqrt{4+4+1}}\right)\frac{2i-2j+k}{\sqrt{4+4+1}} \Rightarrow \frac{10}{9}(2i-2j+k)$$

Note: at Dot Product $i.i = 1,\ j.j = 1,\ k.k = 1$ while $i.j = 0,\ j.k = 0,\ k.i = 0$

Note: at cross Product $i\times j = k,\ j\times k = i,\ k\times i = j$ while $i\times i = 0,\ j\times j = 0,\ k\times k = 0$

Note: the cross product of any two vector result in a vector perpendicular to them such as:
$if\ \vec{V_1}\times\vec{V_2} = \vec{V} \Rightarrow \vec{V}\perp\vec{V_1}\ and\ \vec{V}\perp\vec{V_2}$

Note: $if\ \vec{V_1}.\vec{V_2} = 0$ **this means that** $\vec{V_1}\perp\vec{V_2}$ **(the two vectors are orthogonal)**

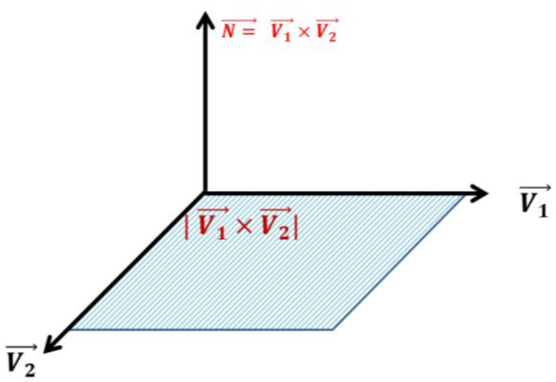
Note: $if\ \vec{V_1}\times\vec{V_2} = 0$ **this means that** $\vec{V_1}//\vec{V_2}$ **(the two vectors are parallel)**

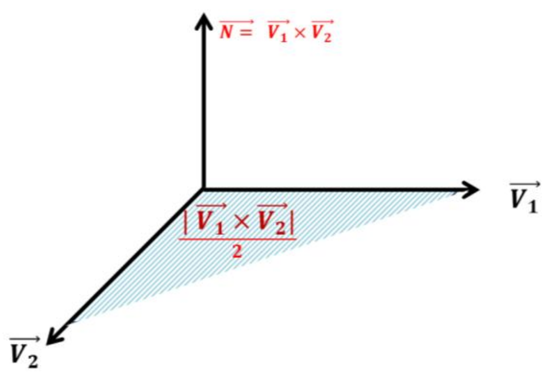
Note: Area of a Parallelogram between any two vectors $\vec{V_1}\ and\ \vec{V_2}$ **is** $|\vec{V_1}\times\vec{V_2}|$

Note: Area of triangle between any two vectors $\vec{V_1}\ and\ \vec{V_2}$ **is** $|\vec{V_1}\times\vec{V_2}|/2$

Note: The angle between any two vectors $\vec{V_1}\ and\ \vec{V_2}$ **is** $\cos\theta = \frac{\vec{A}\ .\vec{B}}{|\vec{A}|\ |\vec{B}|} \Rightarrow \theta = \cos^{-1}\frac{\vec{A}\ .\vec{B}}{|\vec{A}|\ |\vec{B}|}$

Note: projection of $\vec{A}\ onto\ \vec{B}$ is $proj_{\,B}A = \left(\frac{\vec{A}\ .\vec{B}}{|B|}\right)\frac{\vec{B}}{|B|}$ **is vector projection**





Area of parallelogram between two vectors , **Area of Triangle between two vectors**

Let $q(1,4,3)$, $p(5,2,1)$. Find	
Vector between two points $\overrightarrow{qp}$	$\overrightarrow{qp} = (5-1)i + (2-4)j + (1-3)k$ $\Rightarrow \overrightarrow{qp} = 4i - 2j - 2k$
Length (magnitude) distance	$\Rightarrow \lvert\overrightarrow{qp}\rvert = \sqrt{4^2+2^2+2^2}$ $\Rightarrow \lvert\overrightarrow{qp}\rvert = \sqrt{32}$ unit length
Let $\vec{A} = 4i - 2j + k$ and $\vec{B} = 3i + j + 2k$. Find	
Unit vectors of $\vec{A}$ & $\vec{B}$	$\Rightarrow u_A = \frac{\vec{A}}{\lvert\vec{A}\rvert} = \frac{4i-2j+k}{\sqrt{4^2+2^2+1^2}}$, $u_B = \frac{\vec{B}}{\lvert\vec{B}\rvert} = \frac{3i+j+2k}{\sqrt{3^2+1^2+2^2}}$
Dot product $(\vec{A}.\vec{B})$	$\Rightarrow \vec{A}\, dot\, \vec{B} = (4i-2j+k)\cdot(3i+j+2k) \Rightarrow \vec{A}\cdot\vec{B} = 12-2+2$ $\Rightarrow \vec{A}\cdot\vec{B} = 12$
Cross product $(\vec{A}\times\vec{B})$	$\Rightarrow \vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 4 & -2 & 1 \\ 3 & 1 & 2 \end{vmatrix}\begin{matrix} i & j \\ 4 & -2 \\ 3 & 1 \end{matrix}$ $\Rightarrow \vec{A}\times\vec{B} = (-4i+3j+4k) - (-6k+i+8j)$ $\Rightarrow \vec{A}\times\vec{B} = -5i-5j+10k$
Perpendicular Vectors on $\vec{A}$ & $\vec{B}$ (Normal Vector $\vec{N}$)	$\vec{N} = \vec{A}\times\vec{B} = \cdots$ …. ….. $\Rightarrow \vec{N} = \vec{A}\times\vec{B} = -5i-5j+10k$
Test the Vectors $\vec{A}$ & $\vec{B}$ is Parallel or orthogonal or not	$\Rightarrow \vec{A}\cdot\vec{B} =$ $\Rightarrow \vec{A}\times\vec{B} =$
Angle between two vectors	$\cos\theta = \frac{\vec{A}\,.\,\vec{B}}{\lvert\vec{A}\rvert\ \lvert\vec{B}\rvert}$
Area of parallelogram & Area of triangle	$\Rightarrow \vec{A}\times\vec{B} = -5i-5j+10k$
Projection $proj_B A$	$proj_B A = \left(\frac{\vec{A}\,.\vec{B}}{\lvert B\rvert}\right)\frac{\vec{B}}{\lvert B\rvert}$

H.W:1 $A(6,-7,-1)$, $B(2,-3,1)$, $C(4,-5,0)$ are three point: Find

1) The vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$.

2) The unit vectors of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$

3) $\overrightarrow{AB}$ dot $\overrightarrow{BC}$, and $\overrightarrow{CA} \times \overrightarrow{BC}$

4) The angles between $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{BC}$

5) The projection $\overrightarrow{AB}$ on to $\overrightarrow{BC}$ *as a scalar*

6) The projection $\overrightarrow{CA}$ on to $\overrightarrow{BC}$ *as a Vector*

H.W:2 $\vec{A} = 3i - j - 2k$, $\vec{B} = -6i + 2j + 4k$, $\vec{C} = -4i + 2j$: Find

1) The unit vectors of $\vec{A}$, $\vec{B}$, $\vec{C}$

2) $\vec{A}\times\vec{B}$, $\vec{A}\times\vec{C}$, $\vec{B}\times\vec{C}$

3) $\vec{A}.\vec{B}$, $\vec{A}.\vec{C}$, $\vec{B}.\vec{C}$

4) The angles between $\vec{A}$ and $\vec{B}$, $\vec{B}$ and $\vec{C}$

5) The projection $\vec{A}$ on to $\vec{B}$ *as a scalar*

6) Find the area of Parallelogram and area of triangle between $\vec{A}$ *and* $\vec{B}$. ,

Examples:

Ex. (1): Find the angle between **A** = 2i + 2j -k and **B** = 6i - 3j + 2k ?

Solution:

A . B = AB cos θ, $A = |\mathbf{A}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3$

$$B = |\mathbf{B}| = \sqrt{(6)^2 + (-3)^2 + (2)^2} = 7$$

A. B = (2)(6) + (2)(-3) + (-1)(2) = 4,

Then $\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{4}{(3)(7)} = 0.1905, \qquad \therefore\ \theta \approx 79°$

Example 1: Find the angle between the two vectors 2i + 3j + k, and 5i -2j + 3k.

Solution:

The two given vectors are:

$\vec{a}$ = 2i + 3i + k, and $\vec{b}$ = 5i -2j + 3k

$|a| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

$|b| = \sqrt{5^2 + (-2)^2 + 3^2} = \sqrt{25 + 4 + 9} = \sqrt{38}$

Using the dot product we have $\vec{a} . \vec{b}$ = 2.(5) + 3.(-2) + 1.(3) = 10 - 6 + 3 = 7

$$Cos\theta = \frac{a.b}{|a|.|b|}$$

$$= \frac{7}{\sqrt{14}.\sqrt{38}}$$

$$= \frac{7}{2.\sqrt{7 \times 19}}$$

$$= \frac{7}{2\sqrt{133}}$$

$$\theta = Cos^{-1}\frac{7}{2\sqrt{133}}$$

$\theta = Cos^{-1} 0.304 = 72.3°$

Answer: Therefore the angle between the vectors is 72.3°

Example 2: Find the cross product of two vectors $\vec{a}$ = (3,4,5) and $\vec{b}$ = (7,8,9)

Solution:

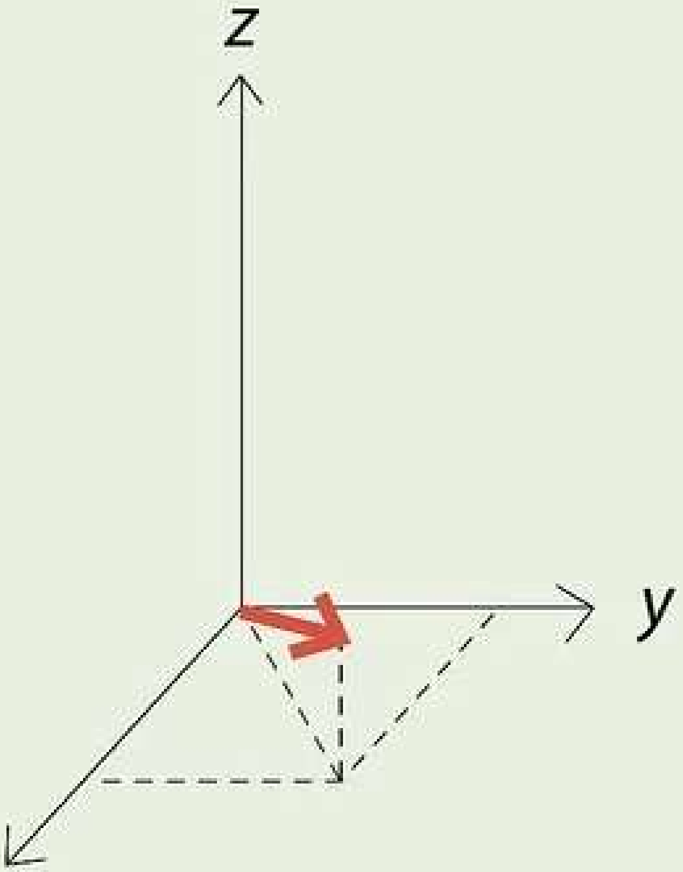
The cross product is given as,

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix}$$

$= [(4\times9)-(5\times8)]\,\hat{i} - [(3\times9)-(5\times7)]\hat{j} + [(3\times8)-(4\times7)]\,\hat{k}$

$= (36-40)\hat{i} - (27-35)\hat{j} + (24-28)\,\hat{k} = -4\hat{i} + 8\hat{j} - 4\hat{k}$

Answer: Therefore, $\vec{a} \times \vec{b} = -4\hat{i} + 8\hat{j} - 4\hat{k}$



$$\mathbf{a = 2\hat{i} + 2\hat{j} + 1\hat{k}}$$

$$\mathbf{\hat{a} = ?}$$

$$|a| = \sqrt{2^2 + 2^2 + 1^2}$$

$$= \sqrt{9} = 3$$

$$\hat{a} = \frac{2\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{1\hat{k}}{3}$$