an object has a geometrical structure such as that of being a "point", a "line", a "curve", a "collection of curves", or a "region of points".

1.2 Translations

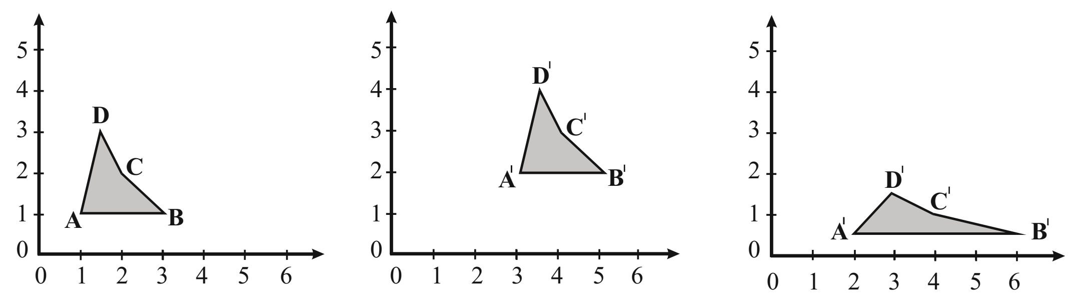
A *translation* is a transformation which maps a point $\mathbf{P}(x, y)$ to a point $\mathbf{P}'(x', y')$ by adding a constant amount to each coordinate so that

$$x' = x + h, \quad y' = y + k \ ,$$

for some constants h and k. The translation has the effect of moving $\mathbf{P}$ in the direction of the x-axis by h units, and in the direction of the y-axis by k units. If $\mathbf{P}$ and $\mathbf{P}'$ are written as row vectors, then

$$(x', y') = (x, y) + (h, k) \ .$$

To translate an object it is necessary to add the vector (h, k) to every point of that object. The translation is denoted $\mathsf{T}\,(h, k)$. A translation can also be executed using matrix addition if (x, y) is represented as the row matrix $\begin{pmatrix} x & y \end{pmatrix}$.



(a) Original quadrilateral (b) Effect of a translation (c) Effect of a scaling

Figure 1.1

Example 1.8

Consider a quadrilateral with vertices $\mathbf{A}(1, 1)$, $\mathbf{B}(3, 1)$, $\mathbf{C}(2, 2)$, and $\mathbf{D}(1.5, 3)$. Applying the translation $\mathsf{T}\,(2, 1)$, the images of the vertices are

$$\begin{aligned} \mathbf{A}' &= (1, 1) + (2, 1) = (3, 2) \ , \\ \mathbf{B}' &= (3, 1) + (2, 1) = (5, 2) \ , \\ \mathbf{C}' &= (2, 2) + (2, 1) = (4, 3) \ , \text{ and} \\ \mathbf{D}' &= (1.5, 3) + (2, 1) = (3.5, 4) \ . \end{aligned}$$

Figure 1.1 shows (a) the original, and (b) the translated quadrilateral.

Definition 1.9

The transformation which leaves all points of the plane unchanged is called the *identity transformation* and denoted I. The *inverse transformation* of L, denoted L^{-1}, is the transformation such that (i) L^{-1} maps every image point $L(\mathbf{P})$ back to its original position $\mathbf{P}$, and (ii) L maps every image point $L^{-1}(\mathbf{P})$ to $\mathbf{P}$. Inverse transformations will be discussed further in Section 2.5.1.

Example 1.10

Consider the translation $\mathsf{T}(h,k)$ which maps a point $\mathbf{P}(x,y)$ to $\mathbf{P}'(x+h,y+k)$. The transformation T^{-1} required to map $\mathbf{P}'$ back to $\mathbf{P}$ is the inverse translation $\mathsf{T}(-h,-k)$. For instance, applying $\mathsf{T}(-2,-1)$ to the point $\mathbf{A}'$ of Example 1.8 gives $(3,2)+(-2,-1)=(1,1)$, and hence maps $\mathbf{A}'$ back to $\mathbf{A}$. The reader can check that the same translation returns the other images to their original locations.

Exercise 1.7

(a) Apply the translation $\mathsf{T}(3,-2)$ to the quadrilateral of Example 1.8, and make a sketch of the transformed quadrilateral.

(b) Determine the inverse transformation of $\mathsf{T}(3,-2)$. Apply the inverse to the transformed quadrilateral to verify that the inverse returns the quadrilateral to its original position.

1.3 Scaling about the Origin

A *scaling about the origin* is a transformation which maps a point $\mathbf{P}(x,y)$ to a point $\mathbf{P}'(x',y')$ by multiplying the x and y coordinates by non-zero constant *scaling factors* s_x and s_y, respectively, to give

$$x' = s_x x \quad \text{and} \quad y' = s_y y \; .$$

A scaling factor s is said to be an *enlargement* if $|s| > 1$, and a *contraction* if $|s| < 1$. A scaling transformation is said to be *uniform* whenever $s_x = s_y$. By representing a point (x,y) as a row matrix $\begin{pmatrix} x & y \end{pmatrix}$, the scaling transformation can be performed by a matrix multiplication

$$\mathbf{P}' = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} = \begin{pmatrix} s_x x & s_y y \end{pmatrix} \; .$$

The matrix

$$\mathsf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$$

is called the *scaling transformation matrix*.

Example 1.11

To apply the scaling transformation $\mathsf{S}(2, 0.5)$ to the quadrilateral of Example 1.8, the coordinates of the four vertices of the quadrilateral are represented by the rows of the 4×2 matrix

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix},$$

and multiplied by the scaling transformation matrix

$$\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 2 & 0.5 \\ 6 & 0.5 \\ 4 & 1 \\ 3 & 1.5 \end{pmatrix}.$$

The rows of the resulting matrix are the coordinates of the images of the vertices. The original quadrilateral and its scaled image are shown in Figures 1.1(a) and (c). The quadrilateral is scaled by a factor 2 in the x-direction and by a factor 0.5 in the y-direction.

Remark 1.12

The quadrilateral of Example 1.11 has experienced a translation due to the fact that scaling transformations are performed about the origin **O**. (Scalings about an arbitrary point are considered in Section 2.4.2.) The true effect of a scaling about the origin is to scale the position vectors $\overrightarrow{\mathbf{OP}}$ of each point **P** in the plane. For instance, in Example 1.11 vectors $\overrightarrow{\mathbf{OA}}$, $\overrightarrow{\mathbf{OB}}$, $\overrightarrow{\mathbf{OC}}$, and $\overrightarrow{\mathbf{OD}}$ have been scaled by the factors 2 and 0.5 in the x- and y-directions as shown in Figure 1.2. Since the positions of all four points **A**, **B**, **C**, and **D** have changed, there is a combined effect of scaling and translating of the object. The origin is the only point unaffected by a scaling about the origin.

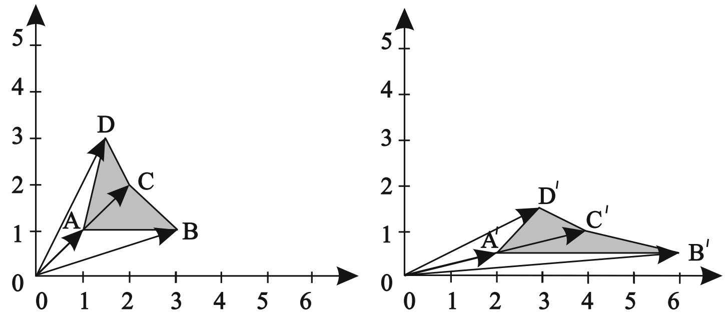


Figure 1.2 Effect of scaling on position vectors

EXERCISES

1.8. Apply the scaling transformation $\mathsf{S}(-1, 1)$ to the quadrilateral of Example 1.8. Describe the effect of the transformation.

1.9. Show that the inverse transformation $\mathsf{S}(s_x, s_y)^{-1}$ of a scaling $\mathsf{S}(s_x, s_y)$ (with $s_x \neq 0$ and $s_y \neq 0$) is the scaling $\mathsf{S}(1/s_x, 1/s_y)$.

1.4 Reflections

Two effects which are commonly used in CAD or computer drawing packages are the horizontal and vertical "flip" or "mirror" effects. Pictures which have undergone a horizontal or vertical flip are shown in Figure 1.3(a). A flip of an object is obtained by applying a transformation known as a *reflection*. Consider a fixed line ℓ in the plane. The reflected image of a point $\mathbf{P}$, a distance d from ℓ, is determined as follows. If $d = 0$ then $\mathbf{P}$ is a point on ℓ and the image is $\mathbf{P}$. Otherwise, take the unique line ℓ_1 through $\mathbf{P}$ and perpendicular to ℓ. Then, as showed in Figure 1.3(b), there are two distinct points on ℓ_1, $\mathbf{P}$ and $\mathbf{P'}$, which are a distance d away from ℓ. The point $\mathbf{P'}$ is the required image of $\mathbf{P}$.

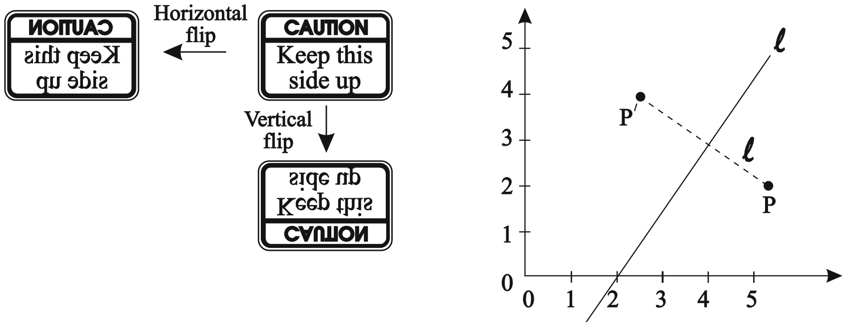
It is easily verified that the reflection R_x in the x-axis is the transformation $L(x, y) = (x, -y)$, and the reflection R_y in the y-axis is $L(x, y) = (-x, y)$. The reflection R_x can be computed by the matrix multiplication

$$\mathsf{R}_x \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x & -y \end{pmatrix} ,$$

and R_y by

$$\mathsf{R}_y \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x & y \end{pmatrix} .$$

The reflection R_y was encountered in Exercise 1.8. Reflections in arbitrary lines are discussed in Section 2.5.3.



(a) Horizontal and vertical flips (b) Reflection in the line ℓ

Figure 1.3

EXERCISES

1.10. Apply the reflection R_x to the quadrilateral of Example 1.8.

1.11. Verify that $\mathsf{R}_x = \mathsf{S}(1, -1)$ and $\mathsf{R}_y = \mathsf{S}(-1, 1)$.

1.12. Show that the inverse of R_x is R_x, that is, $\mathsf{R}_x^{-1} = \mathsf{R}_x$. Similarly, show that $\mathsf{R}_y^{-1} = \mathsf{R}_y$.

1.5 Rotation about the Origin

A *rotation* about the origin through an angle θ has the effect that a point $\mathbf{P}(x, y)$ is mapped to a point $\mathbf{P}'(x', y')$ so that the initial point $\mathbf{P}$ and its image point $\mathbf{P}'$ are the same distance from the origin, and the angle between lines $\overline{\mathbf{OP}}$ and $\overline{\mathbf{OP}'}$ is θ. There are two possible image points which satisfy these properties depending on whether the rotation is carried out in a clockwise or anticlockwise direction. It is the convention that a positive angle θ represents an *anticlockwise* direction so that a $\pi/2$ rotation about the origin maps points on the x-axis to points on the y-axis.

Referring to Figure 1.4, let $\mathbf{P}'(x', y')$ denote the image of a point $\mathbf{P}(x, y)$ following a rotation about the origin through an angle θ (in an anticlockwise direction). Suppose the line $\overline{\mathbf{OP}}$ makes an angle ϕ with the x-axis, and that $\mathbf{P}$ is a distance r from the origin. Then $(x, y) = (r\cos\phi, r\sin\phi)$. $\mathbf{P}'$ makes an angle $\theta + \phi$ with the x-axis, and therefore $(x', y') = (r\cos(\theta+\phi), r\sin(\theta+\phi))$. The addition formulae for trigonometric functions yield

$$\begin{aligned} x' &= r\cos(\theta+\phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi = x\cos\theta - y\sin\theta\ , \text{ and} \\ y' &= r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi = x\sin\theta + y\cos\theta\ . \end{aligned}$$

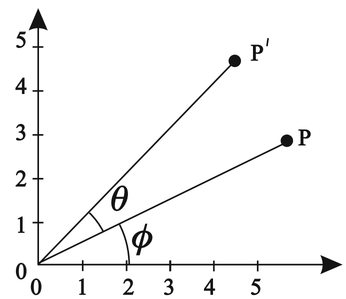


Figure 1.4 Rotation of a point **P** about the origin

The coordinates (x', y') can be obtained from (x, y) by the matrix multiplication

$$\mathbf{P}' = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta \end{pmatrix} .$$

The matrix

$$\mathsf{Rot}\,(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is called the *rotation matrix*.

Example 1.13

The rotation matrices of rotations about the origin through $\pi/2$, π, and $3\pi/2$ radians are

$$\mathsf{Rot}\,(\pi/2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathsf{Rot}\,(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathsf{Rot}\,(3\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Example 1.14

Applying the rotation $\mathsf{Rot}\,(\pi/2)$ to the quadrilateral of Example 1.8, gives the points

$$\begin{pmatrix} \mathbf{A}' \\ \mathbf{B}' \\ \mathbf{C}' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 2 \\ 1.5 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 3 \\ -2 & 2 \\ -3 & 1.5 \end{pmatrix} .$$

The image of the quadrilateral is shown in Figure 1.5.

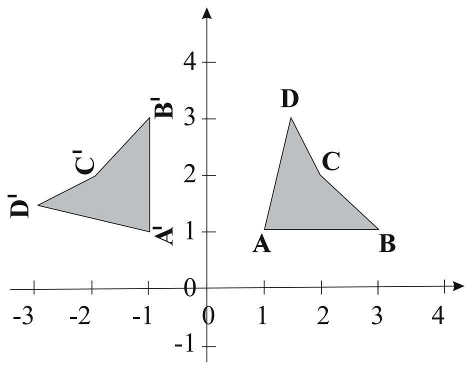


Figure 1.5 Rotation of the quadrilateral about the origin through $\pi/2$

EXERCISES

1.13. Apply rotations about the origin through the angles $\pi/3$, $2\pi/3$, and $\pi/4$ to the triangle with vertices $\mathbf{P}(1,1)$, $\mathbf{Q}(3,1)$, and $\mathbf{R}(2,2)$. Sketch the resulting triangles.

1.14. Show that $\mathsf{Rot}\,(\theta)^{-1} = \mathsf{Rot}\,(-\theta)$.

1.15. Do the transformations $\mathsf{Rot}\,(\pi/2)$ and R_y have the same effect?

1.6 Shears

Given a fixed direction in the plane specified by a unit vector $\mathbf{v} = (v_1, v_2)$, consider the lines ℓ_d with direction $\mathbf{v}$ and a distance d from the origin as shown in Fig.ure 1.6. A *shear about the origin of factor* r *in the direction* $\mathbf{v}$ is defined to be the transformation which maps a point $\mathbf{P}$ on ℓ_d to the point $\mathbf{P}' = \mathbf{P} + rd\mathbf{v}$. Thus the points on ℓ_d are translated along ℓ_d (that is, in the direction of $\mathbf{v}$) through a distance of rd. Shears can be used to obtain italic fonts from normal fonts (see Section 8.1.3).

Example 1.15

To determine a shear in the direction of the x-axis with factor r, let $\mathbf{v} = (1,0)$. The line in the direction of $\mathbf{v}$ through an arbitrary point $\mathbf{P}(x_0, y_0)$ has the equation $y = y_0$. The line is a distance y_0 from the origin. Thus $\mathbf{P}$ is mapped to $\mathbf{P}'(x_0 + ry_0, y_0)$ and hence

$$\begin{pmatrix} x' & y' \end{pmatrix} = \begin{pmatrix} x_0 + ry_0 & y_0 \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$