Applications of Partial Derivatives :-

تطبيقات المشتقات الجزئية :-

① Tangent plane to Surface :-

المستوي المماس للسطح :-

From the partial derivatives topic, we learned that F_x & F_y are slopes of traces. Here we like to extend this idea out a little and start to think of the plane that is tangent to the surface as a point.

IF $F(x,y,z)=0$ is a plane Function, Then The general equation of a plane is given by:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Where (x_0, y_0, z_0) is a point That is on The plane. We can re-write the eq= of the plane as Follows,

$$z-z_0 = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0)$$

let $A=-\frac{a}{c}$ &
$B=-\frac{b}{c}$

Email (hussein.kadhim@uomus.edu.iq)

$\therefore \; z - z_0 = A(x - x_0) + B(y - y_0)$

We need to determine values for $\underline{\underline{A}}$ & $\underline{\underline{B}}$.

* Lets assume $y = y_0$

$\therefore \; z - z_0 = A(x - x_0) \longleftarrow$ Eq= of line and must be tangent to the surface at (x_0, y_0), so it is part of tangent plane!!

$$\therefore \; A = \frac{z - z_0}{x - x_0} = \text{slope at } (x_0, y_0)$$

$$\therefore \; A = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$$

* Lets assume $x = x_0$

$\therefore \; z - z_0 = B(y - y_0) \longleftarrow$ Eq= of line and must be tangent to the surface at (x_0, y_0), so it is part of tangent plane!!

$$\therefore \; B = \frac{z - z_0}{y - y_0} = \text{slope at } (x_0, y_0)$$

$$\therefore \; B = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0)$$

$\therefore$ The equation of the tangent plane to the surface is given by;

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$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

or

put $z_0 = f(x_0,y_0)$

$$\therefore \; z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

معادلة المستوي المماس للسطح

Ex1 Find the equation of the tangent plane to $z = \ln(2x + y)$ at $(-1,3)$

Solution

To Find the eq. of the tangent, then we need to get $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ & $f(x_0,y_0)$

$\therefore \; f(x,y) = z = \ln(2x + y)$

$* \; f(x_0,y_0)\Big|_{(-1,3)} = z_0 = \ln(2*(-1) + 3 = \ln(1) = \boxed{\text{Zero}}$

$* \; f_x = \dfrac{\partial f}{\partial x} = \dfrac{2}{2x+y} \Rightarrow f_x(x_0,y_0)\Big|_{(-1,3)} = \dfrac{2}{2(-1)+3} = \boxed{2}$

$* \; f_y = \dfrac{\partial f}{\partial y} = \dfrac{1}{2x+y} \Rightarrow f_y(x_0,y_0)\Big|_{(-1,3)} = \dfrac{1}{2(-1)+3} = \boxed{1}$

$\therefore$ The eq. of the tangent plane is:

$z - 0 = 2(x - (-1)) + (1)(y - 3) \Rightarrow \boxed{z = 2x + y - 1}$ Ans

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Linear Approximation التقريب الخطي

One nice use of tangent planes is they give us a way to approximate a surface near a point. In other word, As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point.

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Ex)

Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4, 3)$?

Solution)

Here, we are really asking for the tangent plane, so let's find it.

$$f(x,y) = 3 + \frac{x^2}{16} + \frac{y^2}{9} \Rightarrow f(-4,3) = 3 + \frac{(-4)^2}{16} + \frac{3^2}{9} = \boxed{5}$$

$$f_x(x,y) = \frac{x}{8} \Rightarrow f_x(-4,3) = \boxed{-\frac{1}{2}}$$

$$f_y(x,y) = \frac{2y}{9} \Rightarrow f_y(-4,3) = \boxed{\frac{2}{3}}$$

$$\therefore L(x,y) = 5 + (-\tfrac{1}{2})(x-(-4)) + \tfrac{2}{3}(y-3)$$

$$\boxed{L(x,y) = 5 - \tfrac{1}{2}(x+4) + \tfrac{2}{3}(y-3)} \quad \text{Ans}$$

Email (hussein.kadhim@uomus.edu.iq)

Ex// Find the equation of the tangent plane for the surface $z = y\cos(x-y)$ at $(2,2,2)$?

Solution/

$(x_0, y_0, z_0) = (2,2,2)$, $f(x,y) = z = y\cos(x-y)$

* The general form of tangent plane to surface eq. is :

$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$$f(x_0,y_0)\Big|_{(2,2,2)} = z_0 = \boxed{2}$$

* $f_x(x,y) = y(-\sin(x-y) * 1) = -y\sin(x-y)$

$f_x(x_0,y_0) = f_x(2,2) = -2\sin\overset{\text{zero}}{(2-2)} = \boxed{\text{zero}}$

* $f_y(x,y) = y(-\sin(x-y) * (-1)) + \cos(x-y) * 1$

$f_y(x,y) = y\sin(x-y) + \cos(x-y)$

$f_y(x_0,y_0) = f_y(2,2) = 2\sin\overset{0}{(2-2)} + \cos\overset{1}{(2-2)} = \boxed{1}$

$\therefore z = 2 + \text{zero}(x-2) + 1(y-2) = \cancel{2} + y - \cancel{2}$

$\therefore \boxed{z = y}$ Ans

Normal Line to Surface :- الخط العمودي على السطح

If we recall the equation of a plane that has a point (x_0, y_0, z_0) with a normal vector $\vec{N} = ai + bj + ck$ which is given by;

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad \text{(From vectors lecture)}$$

Fact :- حقيقة

The gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal (متعامدة) to the level surface $f(x, y, z) = K$ at the point (x_0, y_0, z_0).

From this Fact, the gradient vector is always normal to the surface at a point !!

Note] The gradient vector is $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$ $\vec{N}$

The tangent plane to the surface given by $f(x, y, z) = K$ at (x_0, y_0, z_0) has the equation;

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

General Form of Tangent Plane to Surface

And the eq. of the normal line are

$$x = x_0 + f_x(x_0, y_0, z_0)\,t$$
$$y = y_0 + f_y(x_0, y_0, z_0)\,t$$
$$z = z_0 + f_z(x_0, y_0, z_0)\,t$$

معادلة الخط المتقيم العمودي على المستوي

Email (hussein.kadhim@uomus.edu.iq)

Ex/ Find the tangent plane equation & the normal line on the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $(1,2,4)$?

Solution

$(x_0,y_0,z_0)=(1,2,4)$

$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$

* $f_x(x_0,y_0,z_0)=2x \Longrightarrow f_x(1,2,4)=2\times1=\boxed{2}$
* $f_y(x_0,y_0,z_0)=2y \Longrightarrow f_y(1,2,4)=2\times2=\boxed{4}$
* $f_z(x_0,y_0,z_0)=1 \Longrightarrow f_z(1,2,4)=\boxed{1}$

$\therefore 2(x-1)+4(y-2)+1(z-4)=0$

or $\boxed{2x+4y+z=14} \longleftarrow$ Ans Eq. of tangent plane

The eqs of the normal line are:

$$\left.\begin{aligned} x&=x_0+f_x(1,2,4)\,t=1+2t\\ y&=y_0+f_y(1,2,4)\,t=2+4t\\ z&=z_0+f_z(1,2,4)\,t=4+t \end{aligned}\right\} \longleftarrow$$ Ans Eqs of normal line

Tangent Line to Curve :- ايجاد ميل المنحني

To Find a tangent line equation, we need a slope and a point (x, y) then by using the equation

$$y - y_1 = m(x - x_1) \quad , \quad m = \text{slope}$$

Ex) Find the equation of a tangent line to the curve $y = 3x^4 - 5x^3 + 6x + 8$ at $x = 1$?

Solution

In this case, we need to Find the point (x, y) that has $x = 1$, by plugging $x = 1$ into the eq =

$$y = 3(1)^4 - 5(1)^3 + 6(1) + 8 = \boxed{12}$$

$\therefore$ the point is $(1, 12)$

Now to find the slope, we need to Find the 1st derivative $\frac{dy}{dx}$ & plugging the point in

$$\frac{dy}{dx} = 3*4x^3 - 5*3x^2 + 6 \Longrightarrow \left.\frac{dy}{dx}\right|_{x=1} = \text{slope} = 12(1)^3 - 15(1)^2 + 6$$

$$\therefore \frac{dy}{dx} = \text{slope} = 12 - 15 + 6 = \boxed{3} = m$$

$$\therefore y - y_1 = m(x - x_1) \Longrightarrow y - 12 = 3(x - 1)$$

$$\therefore \boxed{y = 3x + 9} \quad \text{Ans.}$$

Note/ To Find the equation of tangent line to curve, there is another useful form $y = mx + B$ which we can use as well

to solve the previous example by using this form, we need to find the value of B, by plugging in m & (x,y) as follows

$12 = 3(1) + B \Longrightarrow B = 12 - 3 = \boxed{9}$

$\therefore \boxed{y = 12x + 9} \longleftarrow$ same answer

Ex/ Find the equation of tangent line to the curve $y = 2x^3 - 6x^2 + 5x + 1$ at $x = 2$?

Solution

As we learned from the previous example, we need to find a point (x,y) & slope (m)

$y = 2(2)^3 - 6(2)^2 + 5(2) + 1$
$= 16 - 24 + 10 + 1$
$y = \boxed{3}$

$\therefore$ The point is $(2,3)$

Now we need to find a slope of the eqs at $(2,3)$

Email (hussein.kadhim@uomus.edu.iq)

The slope is the 1st derivative $\left(\frac{dy}{dx}\right)$

$\therefore \frac{dy}{dx} = y' = 6x^2 - 12x + 5$

$\left.\frac{dy}{dx}\right|_{x=2} = 6(2)^2 - 12(2) + 5 = 24 - 24 + 5 = \boxed{5}$

$\therefore \boxed{\text{slope} = m = 5}$

To find the eqs of the tangent line plug in m & (2,3) into the eqs $y - y_1 = m(x - x_1)$

$\therefore y - 3 = 5(x-2) \Rightarrow y - 3 = 5x - 10$

$\therefore \boxed{y = 5x - 7}$ Ans

Ex) Find a tangent line to the equation $x^2 + 3xy + y^3 = 15$ at $x = 1$

Solution)

First of all, we need to find the point (x, y)

$\therefore (1)^2 + 3(1)y + y^3 = 15$

$1 + 3y + y^3 = 15 \Rightarrow y^3 + 3y - 14 = 0$

To find the value of y, we need to solve this equation, the easiest way is by trying to plug in values to y & see the one that satisfy it.

lets put the eqs in this form

$y^3 + 3y = 14$

y	y^3+3y
0	0
1	4
$\boxed{2}$	$\boxed{14}$ ← o.k

So the point is (1,2) and the slope is m which can be found by drive the eqs & find $\frac{dy}{dx}$.

$\therefore x^2 + 3xy + y^3 = 15$ (drive it !!)

$$2x + 3x \cdot \frac{dy}{dx} + y \times 3 + 3y^2 \times \frac{dy}{dx} = 0$$

$$2x + 3x\frac{dy}{dx} + 3y + 3y^2\frac{dy}{dx} = 0$$

(plug in (1,2) !!)

$$2(1) + 3(1)\frac{dy}{dx} + 3(2) + 3(2)^2\frac{dy}{dx} = 0$$

$$2 + 3\frac{dy}{dx} + 6 + 12\frac{dy}{dx} = 0$$

$$8 + 15\frac{dy}{dx} = 0 \longrightarrow \boxed{\frac{dy}{dx} = \frac{-8}{15} = m}$$

$\therefore$ From the found point & slope, we can find the tangent equation, as follows;

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{-8}{15}(x-1) \Rightarrow y - 2 = -\frac{8}{15}x + \frac{8}{15}$$

$$\therefore y = -\frac{8}{15}x + \frac{8}{15} + 2 \Rightarrow \boxed{y = -\frac{8}{15}x + \frac{38}{15}}$$ Ans

Question for discussion:

Find the equation of the tangent line for $f(x) = \frac{1}{2}x^2 - 2x + 7$ that is parallel to the line $2x - y + 3 = 0$?

(Hint: the slope of the tangent line is same the slope of line $2x - y + 3 = 0$)

Email (hussein.kadhim@uomus.edu.iq)

Normal Plane to Curve معادلة المستوي العمودي على المنحني

لإيجاد معادلة المستوي العمودي على المنحني، نحتاج أن نعرف النقطة التي عندها يحدث التقاطع والتي عادةً ما تعطى بالسؤال وكذلك يجب أن تكون لدينا معرفة بالمعادلة العامة للمستوي ويجب إيجاد الفكتر (المتجه) العمودي على المنحني ($\vec{N}$)

To write the eq. of a plane to a curve at point (x_0, y_0, z_0), we need $\vec{N} = ai + bj + ck$ of the curve & use it to yield;

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Similar to what we study in the topic of "Normal line to surface"

Ex) Find a normal plane equation to the helix $\vec{r}(t) = \overset{x}{\cos t}\, i + \overset{y}{\sin t}\, j + \overset{z}{t}\, K$ at $(0, 1, \pi/2)$?

Vector Function

(Solution)

From the Vector function $\vec{r}(t)$ we can find $\vec{N}$ by find $\nabla F = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$ & $F(x,y,z) = \vec{r}(t)$

$$\nabla F = \overset{a}{-\sin t}\, i + \overset{b}{\cos t}\, j + \overset{c}{1}\, K = \vec{N}$$

The equation of normal plane to curve is;

$$\therefore -\sin t(x-0) + \cos t(y-1) + 1(z-\pi/2) = 0$$

To find t we have the point $(0, 1, \pi/2) = (x, y, z)$

$\cos t = 0$ or $\sin t = 1$ or $\boxed{t = \pi/2}$

$\therefore -1(x-0)+0(y-1)+1(z-\pi/2)=0$

$\therefore \boxed{-x+z-\pi/2=0}$ **Ans**

Directional Derivatives:- المشتقة الاتجاهية

The directional derivative is defined as;

$$D_u f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

where P_0 is (x_0, y_0, z_0)
$\vec{u}$ is a unit vector

Ex) Find the directional derivative of $f(x,y,z)=xy^2z^3$ at the point $P(3,2,1)$ in the direction toward point $Q(5,3,2)$?

Solution:

First we need to find the unit vector $\vec{u}$ from the two points P & Q as follows;

$\vec{PQ} = (5-3)i+(3-2)j+(2-1)k = \boxed{2i+j+k}$

$\vec{u} = \dfrac{\vec{PQ}}{|\vec{PQ}|} = \dfrac{2i+j+k}{\sqrt{(2)^2+(1)^2+(1)^2}} = \boxed{\dfrac{1}{\sqrt{6}}(2i+j+k)}$

$\boxed{\nabla f = \dfrac{\partial f}{\partial x}i + \dfrac{\partial f}{\partial y}j + \dfrac{\partial f}{\partial z}k} = y^2z^3 i + 2xyz^3 j + 3xy^2z^2 k$

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$\nabla f)_{(3,2,1)} = (2^2 \times 1^3)i + (2 \times 3 \times 2 \times 1^3)j + (3 \times 3 \times 2^2 \times 1^2)K$

$\boxed{\nabla f)_{(3,2,1)} = 4i + 12j + 36K}$

$\therefore D_u f)_P = (\nabla f)_P \cdot \vec{u}$

$= (4i + 12j + 36K) \cdot \frac{1}{\sqrt{6}}(2i + j + K)$

$= \frac{1}{\sqrt{6}}(4 \times 2 + 12 \times 1 + 36 \times 1)$

$= \frac{1}{\sqrt{6}}(8 + 12 + 36) = \boxed{\frac{56}{\sqrt{6}}}$ Ans

Ex) Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = 3i - 4j$?

Solution)

$\boxed{\vec{u} = \frac{\vec{v}}{|\vec{v}|}} = \frac{3i + 4j}{\sqrt{3^2 + 4^2}} = \boxed{\frac{3}{5}i + \frac{4}{5}j}$

$\boxed{\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k} = (e^y - \sin(xy) \times y)i + (xe^y - \sin(xy)x)j + 0K$

$\therefore \nabla f = (e^y - y\sin(xy))i + (xe^y - x\sin(xy))j$

$\nabla f)_{(2,0)} = (e^0 - 0\sin(0))i + (2e^0 - 2\sin(0))j = \boxed{i + 2j}$

$$D_u f = \nabla f \cdot \vec{u}$$

$$= (i + 2j) \cdot \left(\frac{3}{5} i - \frac{4}{5} j\right)$$

$$= \left(1 \times \frac{3}{5}\right) + \left(2 \times \left(-\frac{4}{5}\right)\right)$$

$$= \frac{3}{5} - \frac{8}{5} = \frac{-5}{5} = \boxed{-1} \quad \text{Ans}$$

Ex) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of the vector $\vec{A} = 2i - 3j + 6k$?

Solution

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2i - 3j + 6k}{\sqrt{2^2 + (-3)^2 + 6^2}} = \boxed{\frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} k}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= (3x^2 - y^2) i + (-2xy) j + (-1) k$$

$$\nabla f)_{(1,1,0)} = (3 \times 1^2 - 1^2) i - (2 \times 1 \times 1) j - k$$

$$= \boxed{2i - 2j - k}$$

$$\therefore D_u f)_{P_0} = \nabla f)_{P_0} \cdot \vec{u} = (2i - 2j - k) \cdot \left(\frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} k\right)$$

$$= \left(2 \times \frac{2}{7}\right) + \left(-2 \times \left(-\frac{3}{7}\right)\right) + \left(-1 \times \frac{6}{7}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \boxed{\frac{4}{7}} \quad \text{Ans}$$

Maxima & Minima of Two Variables Functions

القيم العظمى والصغرى لدالة ذات متغيرين

Theorem: نظرية

If $f(x,y)$ is a function of two independent variables (x,y) and its 1st & 2nd partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x = f_y = 0$ at point (a,b), then,

① f has a local maximum at (a,b) if $f_{xx} < 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

② f has a local minimum at (a,b) if $f_{xx} > 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

③ f has a saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

غير مجدية / غير حاسم

④ The test is inconclusive / doubtful at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b). Need to find some other way !!

Notes

مميز

The expression $f_{xx} f_{yy} - f_{xy}^2$ is called the discriminant of f. To remember it, use,

$$f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Ex) Find the local extreme values of

$f(x,y) = x^2 + y^2$

(Solution)

$f_x = 2x = 0 \longrightarrow \boxed{x=0} = a$

$f_y = 2y = 0 \longrightarrow \boxed{y=0} = b$

$f_{xx} = 2 > 0$

$f_{yy} = 2$

$f_{xy} = 0 \Longrightarrow f_{xy}^2 = 0$

$\therefore f_{xx} f_{yy} - f_{xy}^2 = 2 \times 2 - 0 = 4 > 0$

$\because f_{xx} > 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$, then the point $(0,0)$ is critical point, & the function $f(x,y) = x^2 + y^2$ has local minimum at $(0,0)$.

Ex) Find the extreme value of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$?

(Solution)

$f_x = y - 2x - 2 \longrightarrow f_{xx} = -2 < 0$

$f_x = 0 \longrightarrow y - 2x - 2 = 0$ ——— ①

$f_y = x - 2y - 2 \longrightarrow f_{yy} = -2$

$f_y = 0 \Longrightarrow -2y + x - 2 = 0$ ——— ②

حل المعادلتين ① و ② لإيجاد قيم (x, y)

Email (hussein.kadhim@uomus.edu.iq)

$y - 2x - 2 = 0 \quad — ①$

$-2y + x - 2 = 0 \quad — ② \quad \times 2$

بالجمع

$-3y - 6 = 0 \Longrightarrow y = -\frac{6}{3} = \boxed{-2} = b$

نعوض عن قيمة y في اي معادلة ① أو ② لايجاد قيمة x

$-2 - 2x - 2 = 0 \Longrightarrow -4 - 2x = 0 \longrightarrow x = \boxed{-2} = a$

∴ الدالة تمتلك حالة متطرفة عند هذه النقطة $\therefore (a,b) = (-2,-2)$

ولمعرفة هذه الحالة نقل الفحص التالي:

$\boxed{f_{xy} = 1}$

$f_{xx} f_{yy} - f_{xy}^2 = -2 \times -2 - 1^2 = 4 - 1 = 3 > 0$

$\therefore$ As $f_{xx} < 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$, then the function $f(x,y)$ has a local maximum at $(-2,-2)$.

$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2\times(-2) - 2(-2) + 4$

$= \boxed{8}$

Applications of Partial Derivatives, Tangent Plane to Surface, Normal Line to Surface, angent line to Curve, Normal Plane to Curve, Directional Derivatives, Maxima and Minimaof Two Variables Functions, " نهاية محاضرة --

تطبيقات المشتقات الجزئية، المستوى المماس للسطح، الخط العمودي على السطح، الخط المماس للمنحنى، المستوى العمودي على المنحنى، المشتقات الاتجاهية، القيم العضمى والصغرى للدوال بمتغيرين"--

Email (hussein.kadhim@uomus.edu.iq)