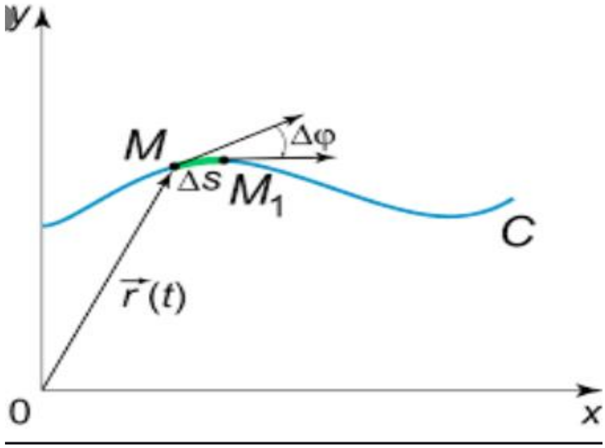
3.8. Curvature, Torsion & binormal vector

Curvature for curves in space



In space there is no natural way to find an angle like ∅ with which to measuring the change in **T** along a differential curve .but we still have **S** ,the directed distance along the curve and can define the curvature to be

$$K=\left|\frac{dT}{dS}\right|$$

OR

$$K=\frac{|vXa|}{|V^3|}$$

Example: Find the curvature of the curve, where a&b$> o$

$$r=(a\cos t\, i + a\sin t\, j + btk)$$

Solution//

$$K=\frac{|vXa|}{|V^3|}$$

$$V=\frac{dr}{dt}=-a\sin t\,i+a\cos t\,j+bk$$

$$a=\frac{dv}{dt}=\ (-a\cos t\,i-a\sin t\,j+0k)$$

$$|V|=\sqrt{(-a\sin t)^2+(a\cos t)^2)+b^{\ 2}}$$

$$=\sqrt{(a^2\sin^2 t)+(a^2\cos^2 t)+b^{\ 2}}$$

$$=\sqrt{a^2(\sin^2 t+\cos^2 t)+b^{\ 2}}$$

$$|V|=\sqrt{a^2+b^2}$$

$$v\,Xa=\begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

=+ab sin t i-ab cos t j+ka^2

$$|vXa|=\sqrt{(ab\sin t)^2+(-ab\cos t)^2+(a^2)^2}$$

$$=\sqrt{a^2b^2\sin^2 t+a^2b^2\cos^2 t+a^4}=\sqrt{a^2b^2+a^4}$$

$$|V|=\sqrt{a^2+b^2}$$

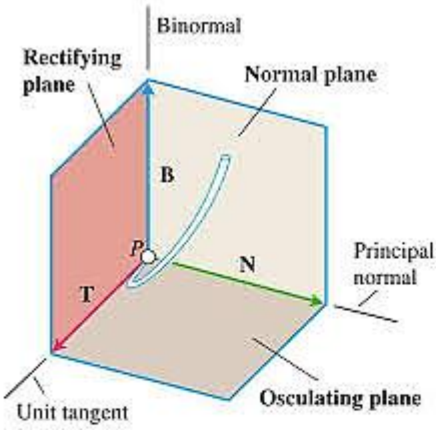
$$|V|^3=(a^2+b^2)^{\frac{3}{2}}$$

$$K=\frac{\sqrt{a^2b^2+a^4}}{(a^{2+}b^2)^{\frac{3}{2}}}$$

Exercise: Find the curvature of the curve,

r= $(c\cos t\,\boldsymbol{i} + c\sin t\,\boldsymbol{j})$

Torsion & binormal vector



B=T x N

Binormal vector is perpendicular to both normal (N)& tangent(T) vector

The torsion $\boldsymbol{\tau} = \left|\frac{dB}{dS}\right|$

It is measure of how mach the curve twists

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2} \qquad \text{if } \mathbf{v\ X\ a} \neq \mathbf{0}$$

Example: Find the torsion of the r= $(\cos t\,\boldsymbol{i} + \sin t\,\boldsymbol{j} + t\boldsymbol{k})$

Solution/

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \, X \, a|^2}$$

$$V = \frac{dr}{dt} = -\sin t \, i + \cos t \, j + k$$

$$a = -\cos t \, i - \sin t j + 0k$$

$$\dot{a} = \sin t \, i - \cos t \, j + 0k$$

$$= \tau = \frac{\begin{vmatrix} -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \\ \sin t & -\cos t & 0 \end{vmatrix}}{\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}^2} = \frac{\cos^2 t + \sin^2 t}{|\sin t \, i - \cos t \, j + k|^2} = \frac{1}{2}$$

Exercise: Find the torsion for the curve r= $(3\sin t \, i + 3\cos t \, j + 4tk)$