



3.6 Integral of vector function and distance along the curve

Integral of vector function

let If $F(t)=f(t)i+g(t)j+h(t)k$ then

$$\int_a^b F(t) dt = \int_a^b f(t)i + \int_a^b g(t)j + \int_a^b h(t)k$$

Example : Find $\int_0^\pi (\cos t i + \sin t j + tk)dt$

Solution//

$$\int_0^\pi \cos t i dt + \int_0^\pi \sin t j dt + \int_0^\pi tk dt = 0i + 2j + \frac{\pi^2}{2}k$$

Exercise: find $\int_0^1 (t^3i + 7j + (t + 1)k)dt$

Distance along the curve

$$\text{Length of curve} = \int_a^b |V| dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$



Example: Find the length of curve of one turn of the
 $\mathbf{r} = (\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k})$.

solution//

$$\text{Length of curve} = \int_a^b |\mathbf{V}| dt$$

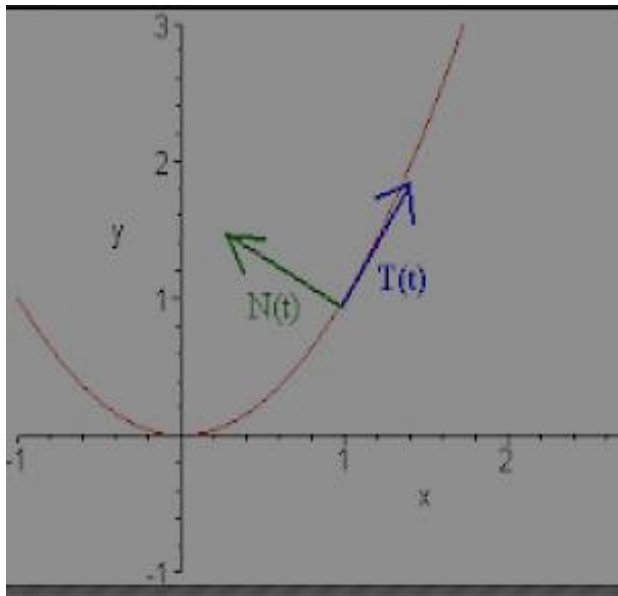
$$\mathbf{V} = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$$

$$|\mathbf{V}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}$$

$$\text{Length of curve} = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}$$

2.7 Unit tangent and normal vector for curve

The Unit Tangent vector





$$S(t) = \int |V| dt$$

$$ds(t) = |V| dt \div dt$$

$$\frac{ds(t)}{dt} = |V|$$

$$\frac{dt}{ds} = \frac{1}{|V|}$$

$$\frac{dr}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds}$$

$$T = \frac{V}{|V|} = \frac{\frac{dr}{dt}}{\left| \frac{dr}{dt} \right|}$$

Example: Find unit tangent vector of the curve

$$r = (\cos t \, i + \sin t \, j)$$

Solution//

$$V = \frac{dr}{dt} = -\sin t \, i + \cos t \, j$$

$$|V| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1$$

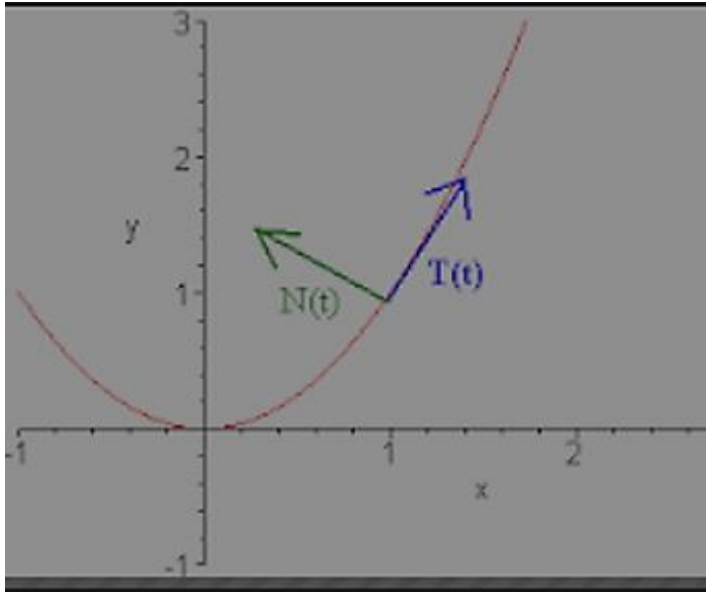
$$T = \frac{V}{|V|} = -\sin t \, i + \cos t \, j$$

Exercise: Find unit tangent vector of the curve

$$r = (2+t)i - (t+1)j + k$$



The Unit Normal vector in plane



$$\mathbf{N} = \frac{\frac{d\mathbf{r}}{dt}}{\left| \frac{d\mathbf{r}}{dt} \right|}$$

Example: Find the tangent and normal vector for curve

$$\mathbf{r} = (\cos 2t \mathbf{i} + \sin 2t \mathbf{j})$$

Solution//

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = -2\sin 2t \mathbf{i} + 2\cos 2t \mathbf{j}$$

$$|\mathbf{V}| = \sqrt{(2\sin 2t)^2 + (2\cos 2t)^2} = 2$$

$$\mathbf{T} = \frac{\mathbf{V}}{|\mathbf{V}|} = -\sin 2t \mathbf{i} + \cos 2t \mathbf{j}$$



$$\frac{dT}{dt} = -2\cos 2t \, i - 2\sin 2t \, j$$

$$\left| \frac{dT}{dt} \right| = 2$$

$$N = \frac{\frac{dT}{dt}}{\left| \frac{dT}{dt} \right|} = -\cos 2t \, i - \sin 2t \, j$$

Example: Find the tangent and normal vector for curve

$$r = (a \cos t^2 \, i + a \sin t^2 \, j + bt^2 \, k)$$

Solution//

$$V = \frac{dr}{dt} = -2at \sin t^2 \, i + 2at \cos t^2 \, j + 2bt \, k$$

$$|V| = 2t\sqrt{a^2 + b^2}$$

$$T = \frac{V}{|V|} = \frac{-a \sin t^2 \, i}{\sqrt{a^2 + b^2}} + \frac{a \cos t^2 \, j}{\sqrt{a^2 + b^2}} + \frac{b \, k}{\sqrt{a^2 + b^2}}$$

$$\frac{-2at \sin t^2 \, i + 2at \cos t^2 \, j + 2bt \, k}{2t\sqrt{a^2 + b^2}}$$

$$\frac{dT}{dt} = \frac{-2at \cos t^2 \, i}{\sqrt{a^2 + b^2}} - \frac{2at \sin t^2 \, j}{\sqrt{a^2 + b^2}} + 0 \, k$$

$$\left| \frac{dT}{dt} \right| = \frac{2at}{\sqrt{a^2 + b^2}}$$

$$N = \frac{\frac{dT}{dt}}{\left| \frac{dT}{dt} \right|} = -\cos t^2 \, i - \sin t^2 \, j$$