Example 1: find the z transform of the following sequences:

A- X(n) = 10 (sin 0.25 π n) . U(n).............(1)

Solution:

If we compare the above sequence with the table in the previous slide then,

$$\sin(an)\,u(n) \quad \underset{z^{-1}}{\overset{z}{\Longleftrightarrow}} \quad \frac{z\,sina}{z^2 - 2zcosa + 1}$$

So, x(z) for the sequence 1 is as follows:

$$\frac{10z\sin(0.25\pi)}{z^2 - 2z\cos(0.25\pi) + 1}$$

ROC = |Z| >1

==

B- $e^{0.1n}$ cos 0.25 π n . u(n)………………2

Now compare the transform of the table as follows:

e^{-an}cos (b n) . u(n) $\overset{z}{\underset{z^{-1}}{\Longleftrightarrow}}$

$$\frac{z\{z - e^{-a}\cos b\}}{z^2 - \{2e^{-a}cosb\}z + e^{-2a}}$$

ROC = |Z|>e^{-a}

Now,

X(z) of the sequence in 1 is as follows:

$$\frac{z\{z - e^{0.1}\cos(0.25\pi)}{z^2 - \{2e^{-0.1}cos0.25\pi\}z + e^{-0.2}}$$

ROC = |Z|>$e^{-0.1}$

C: X(n) = $n\,3^n$ u(n)……………………………………………1

If $n\,3^n$ u(n) $\xleftrightarrow[z^{-1}]{z}$ $\dfrac{az}{(z-a)^2}$ ……………….2

And ROC |Z| > |a|

X(Z) according to the original question 1 then

X(Z) = $\dfrac{3Z}{(Z-3)^2}$

And ROC |Z|> |3|

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D: X(n) = e^{-3n} u(n)

Let us compare this with the standard as follows:

If x(n)= e^{-an} u(n) $\xleftrightarrow[z^{-1}]{z}$ $\dfrac{z}{(z-e^{-a})}$ and ROC : |Z| > e^{-a}

Compare with the D yield: X(Z) = $\dfrac{Z}{(z-e^{-3})}$ and ROC : |Z| >e^{-3}

E: X(n) = n^2 u(n)

From standard

If n^2 u(n) $\overset{Z}{\underset{z^{-1}}{\Longleftrightarrow}}$ $\frac{z(z+1)}{(z-1)^3}$ and ROC: |Z| > 1

Compare with the E yield:

X(Z) = $\frac{z(z+1)}{(z-1)^3}$

ROC: ROC: |Z| > 1

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Z- transform and difference equations:

Solution of difference equation by using z-transform:

The difference linear equation classified as follows:

$$y_{n+1} = y_n + d \qquad \text{(1st order)}$$

$$y_{n+1} = A\ y_n \qquad \text{(1st order)}$$

$$y_{n+2} = y_{n+1} + y_n \qquad \text{(2nd order)}$$

Other examples of linear difference equations are

$$y_{n+2} + 4y_{n+1} - 3y_n = n^2 \qquad \text{(2nd order)}$$

$$y_{n+1} + y_n = n\ 3^n \qquad \text{(1st order)}$$

Examples of non-linear difference equations are

$$\begin{aligned} y_{n+1} &= \sqrt{y_n + 1} \\ y_{n+1}^2 + 2\, y_n &= 3 \\ y_{n+1} y_n &= n \\ \cos(y_{n+1}) &= y_n \end{aligned}$$

We shall not consider the problem of solving non-linear difference equations.

The five linear equations listed above also have **constant coefficients**; for example:

$$y_{n+2} + 4y_{n+1} - 3y_n = n^2$$

has the constant coefficients $1,\ 4,\ -3$.

The (linear) difference equation

$$n\, y_{n+2} - y_{n+1} + y_n = 0$$

has one variable coefficient viz n and so is not classified as a constant coefficient difference equation.

Solution of first order linear constant coefficient difference equations:

Consider the first order difference equation

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

The equation could be solved in a step-by-step or **recursive** manner, provided that y_0 is known because

$$y_1 = 4 + 3y_0 \qquad y_2 = 4 + 3y_1 \qquad y_3 = 4 + 3y_2 \quad \text{and so on.}$$

This process will certainly produce the terms of the solution sequence $\{y_n\}$ but the general term y_n may not be obvious.

So consider

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots \tag{1}$$

with initial condition $y_0 = 1$.

We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty}(y_{n+1} - 3y_n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} - 3\sum_{n=0}^{\infty} y_n z^{-n} = 4\sum_{n=0}^{\infty} z^{-n} \tag{2}$$

The three terms in (2) are clearly recognisable as z-transforms.

The right-hand side is the z-transform of the constant sequence $\{4, 4, \ldots\}$ which is $\dfrac{4z}{z-1}$.

If $Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then

$\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z\,Y(z) - zy_0$ (by the left shift theorem).

Consequently (2) can be written

$$z\,Y(z) - zy_0 - 3\,Y(z) = \frac{4z}{z-1} \qquad (3)$$

Equation (3) is the z-transform of the original difference equation (1). The intervening steps have been included here for explanation purposes but we shall omit them in future. The important point is that (3) is no longer a difference equation. It is an **algebraic** equation where the unknown, $Y(z)$, is the z-transform of the solution sequence $\{y_n\}$.

We now insert the initial condition $y_0 = 1$ and solve (3) for $Y(z)$:

$$\begin{aligned} (z-3)Y(z) - z &= \frac{4z}{(z-1)} \\ (z-3)Y(z) &= \frac{4z}{z-1} + z = \frac{z^2+3z}{z-1} \end{aligned}$$

so $$Y(z) = \frac{z^2+3z}{(z-1)(z-3)} \qquad (4)$$

Thus

$$
\begin{aligned}
Y(z) &= z\frac{(z+3)}{(z-1)(z-3)} \\
&= z\left(\frac{-2}{z-1}+\frac{3}{z-3}\right) \qquad \text{(in partial fractions)}
\end{aligned}
$$

so $\quad Y(z) = \dfrac{-2z}{z-1} + \dfrac{3z}{z-3}$

Now, taking inverse z-transforms, the general term y_n is, using the linearity property,

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\,\mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

The symbolic notation $\mathbb{Z}^{-1}$ is common and is short for 'the inverse z-transform of'.

Using standard z-transforms write down y_n explicitly, where

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\,\mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

Checking the solution:

From this solution (5)

$$y_n = -2 + 3^{n+1}$$

we easily obtain

$$y_0 = -2 + 3 = 1 \qquad \text{(as given)}$$

$$y_1 = -2 + 3^2 = 7$$

$$y_2 = -2 + 3^3 = 25$$

$$y_3 = -2 + 3^4 = 79 \qquad \text{etc.}$$

To solve a linear constant coefficient difference equation, three steps are involved:

1. Replace each term in the difference equation by its z-transform and insert the initial condition(s).
2. Solve the resulting algebraic equation. (Thus gives the z-transform $Y(z)$ of the solution sequence.)
3. Find the inverse z-transform of $Y(z)$.

Example 2:

Solve the difference equation

$$y_{n+1} - y_n = d \qquad n = 0, 1, 2, \ldots \qquad y_0 = a \tag{6}$$

where a and d are constants.
(The solution will give the n th term of an arithmetic sequence with a constant difference d and initial term a.)

If $Y(z) = \mathbb{Z}\{y_n\}$ we obtain the algebraic equation

$$z\,Y(z) - zy_0 - Y(z) = \frac{d \times z}{(z-1)}$$

Note that the right-hand side transform is that of a constant sequence $\{d, d, \dots\}$. Note also the use of the left shift theorem.

Now insert the initial condition $y_0 = a$ and then solve for $Y(z)$:

$$\begin{aligned}(z-1)Y(z) &= \frac{d \times z}{(z-1)} + z \times a \\ Y(z) &= \frac{d \times z}{(z-1)^2} + \frac{a \times z}{z-1}\end{aligned}$$

$$\therefore \quad y_n = dn + a \qquad n = 0, 1, 2, \dots \tag{7}$$

using the known z-transforms of the ramp and unit step sequences. Equation (7) may well be a familiar result to you – an arithmetic sequence whose 'zeroth' term is $y_0 = a$ has general term $y_n = a + nd$.

i.e. $\{y_n\} = \{a,\ a+d, \dots a+nd, \dots\}$

N.B. If the term a is labelled as the **first** term (rather than the zeroth) then

$$y_1 = a,\; y_2 = a + d,\; y_3 - a + 2d,$$

so in this case the n th term is

$$y_n = a + (n-1)d$$

rather than (7).

USE THE RIGHT SHIFT THEOREM TO SOLVE THE FOLLOWING DIFFERENCE EQUATIONS:

The problem just solved was given by (6), i.e.

$$y_{n+1} - y_n = d \qquad \text{with } y_0 = a \qquad n = 0, 1, 2, \ldots$$

We obtained the solution

$$y_n = a + nd \qquad n = 0, 1, 2, \ldots$$

Now consider the problem

$$y_n - y_{n-1} = d \qquad n = 0, 1, 2, \ldots$$

with $y_{-1} = a$.

The only difference between the two problems is that the 'initial condition' in (8) is given at $n = -1$ rather than at $n = 0$. Writing out the first few terms should make this clear.

$$\begin{array}{cc}
(6) & (8) \\
y_1 - y_0 = d & y_0 - y_{-1} = d \\
y_2 - y_1 = d & y_1 - y_0 = d \\
\vdots & \vdots \\
y_{n+1} - y_n = d & y_n - y_{n-1} = d \\
y_0 = a & y_{-1} = a
\end{array}$$

The solution to (8) must therefore be the same as for (6) but with every term in the solution (7) of (6) shifted 1 unit to the left.

Thus the solution to (8) is expected to be

$$y_n = a + (n+1)d \qquad n = -1, 0, 1, 2, \ldots$$

(replacing n by $(n+1)$ in the solution (7)).

Use the right shift theorem of z-transforms to solve with initial condition y_{-1}= a

$$y_n - y_{n-1} = d \qquad n = 0, 1, 2, \ldots$$

(a) Begin by taking the z-transform of (8), inserting the initial condition and solving for Y (z):

We have, for the z-transform of (8)

$$\begin{aligned}
Y(z) - (z^{-1}Y(z) + y_{-1}) &= \frac{dz}{z-1} \qquad \text{[Note that here } dz \text{ means } d \times z\text{]} \\
Y(z)(1 - z^{-1}) - a &= \frac{dz}{z-1} \\
Y(z)\left(\frac{z-1}{z}\right) &= \frac{dz}{(z-1)} + a \\
Y(z) &= \frac{dz^2}{(z-1)^2} + \frac{az}{z-1} \qquad (9)
\end{aligned}$$

The second term of $Y(z)$ has the inverse z-transform $\{a\ u_n\} = \{a, a, a, \ldots\}$.

The first term is less straightforward. However, we have already reasoned that the other term in y_n here should be $(n+1)d$.

(b) Show that the z-transform of $(n+1)d$ is $\frac{dz^2}{(z-1)^2}$. Use the standard transform of the ramp and step:

$$\mathbb{Z}\{(n+1)d\} = d\mathbb{Z}\{n\} + d\mathbb{Z}\{1\}$$

by the linearity property

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{(n+1)d\} &= \frac{dz}{(z-1)^2} + \frac{dz}{z-1} \\ &= dz\left(\frac{1+z-1}{(z-1)^2}\right) \\ &= \frac{dz^2}{(z-1)^2} \end{aligned}$$

So, the final solution is:

$$y_n = (n+1)d + a\,u_n \quad \text{i.e.} \quad y_n = a + (n+1)d \quad n = -1, 0, 1, 2, \ldots$$

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$$\text{solve } y_n - 3y_{n-1} = 4 \quad n = 0, 1, 2, \ldots \quad \text{with } y_{-1} = 1. \tag{10}$$

We have, taking the z-transform of (10),

$$Y(z) - 3(z^{-1}Y(z) + 1) = \frac{4z}{z-1}$$

(using the right shift property and inserting the initial condition.)

$$\therefore \quad Y(z) - 3z^{-1}Y(z) = 3 + \frac{4z}{z-1}$$

$$Y(z)\frac{(z-3)}{z} = 3 + \frac{4z}{z-1} \quad \text{so} \quad Y(z) = \frac{3z}{z-3} + \frac{4z^2}{(z-1)(z-3)}$$

Write the second term as $4z\left(\frac{z}{(z-1)(z-3)}\right)$ and obtain the partial fraction expansion of the bracketed term. Then complete the z-transform inversion.

$$\frac{z}{(z-1)(z-3)} = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z-3}$$

We now have

$$Y(z) = \frac{3z}{z-3} - \frac{2z}{z-1} + \frac{6z}{z-3}$$

so

$$y_n = 3 \times 3^n - 2 + 6 \times 3^n = -2 + 9 \times 3^n = -2 + 3^{n+2} \tag{11}$$