EXAMPLES:

Example1 Find the Z–transform of $2n + 3\sin\frac{n\pi}{4} - 5a^4$

Solution: By linearity property

Linearity: $Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}$

$$Z\left\{2n + 3\sin\frac{n\pi}{4} - 5a^4\right\} = 2Z\{n\} + 3Z\left\{\sin\frac{n\pi}{4}\right\} - 5Z\{a^4\}$$

$$= 2Z\{n\} + 3Z\left\{\sin\frac{n\pi}{4}\right\} - 5a^4Z\{1\}$$

$$= \frac{2z}{(z-1)^2} + \frac{3z\ Sin\frac{\pi}{4}}{z^2-2z\cos\frac{\pi}{4}+1} - \frac{5a^4z}{z-1}$$

$$\because Z\{n\} = \frac{z}{(z-1)^2},\ Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1},\ Z\{1\} = \frac{z}{z-1}$$

$$\therefore Z\left\{2n + 3\sin\frac{n\pi}{4} - 5a^4\right\} = \frac{2z}{(z-1)^2} + \frac{\frac{3z}{\sqrt{2}}}{z^2-\sqrt{2}z+1} - \frac{5a^4z}{z-1}$$

Example2 Find the Z–transform of the sequence $\{4, 8, 16, 32, \dots\}$

Solution: $u_n = 2^{n+2}, \quad n = 0,1,2,3 \dots.$

$$Z\{2^{n+2}\} = Z\{2^2 2^n\}$$
$$= 4\,Z\{2^n\} = \frac{4z}{z-2}, \left|\frac{2}{z}\right| < 1 \ \because Z\{a^n\} = \frac{z}{z-a}, \left|\frac{a}{z}\right| < 1$$

Example3 Find the Z–transform of $(n+1)^2$

Solution: $Z\{(n+1)^2\} = Z\{n^2 + 2n + 1\}$

$$= Z\{n^2\} + 2Z\{n\} + Z\{1\}$$
$$= \frac{z^2+z}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1}$$
$$\because Z\{n^2\} = \frac{z^2+z}{(z-1)^3}, Z\{n\} = \frac{z}{(z-1)^2}, Z\{1\} = \frac{z}{z-1}$$

Example4 Find the Z–transform of

i. $n\cos n\theta$ ii. $\sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$ iii. e^{-2n} iv. $a^{|n|}$

Solution: i. $Z\{\cos n\theta\} = \frac{z\,(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$$\therefore Z\{n\cos n\theta\} = -z\frac{d}{dz}\left[\frac{z\left(z-\cos\theta\right)}{z^2-2z\cos\theta+1}\right]$$

$$\because\ Z\{nu_n\} = -z\frac{d}{dz}Z\{u_n\}$$

$$= -z\left[\frac{-z^2\cos\theta+2z-\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= \frac{z^3\cos\theta-2z^2+z\cos\theta}{(z^2-2z\cos\theta+1)^2}$$

ii. $Z\left\{\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\} = Z\left\{\sin\frac{n\pi}{2}\cos\frac{\pi}{4}+\cos\frac{n\pi}{2}\sin\frac{\pi}{4}\right\}$

$$= \cos\frac{\pi}{4}Z\left\{\sin\frac{n\pi}{2}\right\}+\sin\frac{\pi}{4}Z\left\{\cos\frac{n\pi}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1} + \frac{z\left(z-cos\frac{\pi}{2}\right)}{z^2-2z\,cos\frac{\pi}{2}+1}\right]$$

$$\because Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z}{z^2+1} + \frac{z^2}{z^2+1}\right] = \frac{1}{\sqrt{2}}\left[\frac{z+z^2}{z^2+1}\right]$$

iii. $Z\{e^{-2n}\} = Z\{(e^{-2})^n\}$

$$= \frac{z}{z-e^{-2}} \qquad \because Z\{a^n\} = \frac{z}{z-a}$$

$$= \frac{ze^2}{ze^2-1}$$

iv. $a^{|n|} = \begin{cases} a^{-n}, & n < 0 \\ a^n, & n \geq 0 \end{cases}$

Taking two sided $Z-$ transform: $Z\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}$

$$\therefore Z\{a^{|n|}\} = \sum_{-\infty}^{-1} a^{-n} z^{-n} + \sum_{0}^{\infty} a^n z^{-n}$$

$$= [\ldots + a^3z^3 + a^2z^2 + az] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}, \qquad |az| < 1 \text{ and } \left|\frac{a}{z}\right| < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \qquad |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$= \frac{z(1-az)}{(1-az)(z-a)}, \qquad |a| < |z| < \frac{1}{|a|}$$

$$\Rightarrow Z\{a^{|n|}\} = \frac{z}{(z-a)}, \qquad |a| < |z| < \frac{1}{|a|}$$

Example5 Find the Z–transform of $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$

Taking two sided Z– transform: $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

$$\therefore Z\{u_n\} = \sum_{-\infty}^{-1} 2^n z^{-n} + \sum_{0}^{\infty} 3^n z^{-n}$$

$$= \left[\dots + \frac{z^3}{2^3} + \frac{z^2}{2^2} + \frac{z}{2}\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right]$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{3}{z}}, \qquad \left|\frac{z}{2}\right| < 1 \text{ and } \left|\frac{3}{z}\right| < 1$$

$$= \frac{z}{2-z} + \frac{z}{z-3}, \qquad |z| < |2| \text{ and } |3| < |z|$$

$$= \frac{z(z-3)+z(2-z)}{(2-z)(z-3)}, \qquad |3| < |z| < |2|$$

$$\Rightarrow Z\{u_n\} = \frac{z}{z^2-5z+6}, \qquad 3 < |z| < 2$$

$\therefore$ Z–transform does not exist for $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$ as the set $3 < |z| < 2$ is infeasible.

Example6 Find the Z–transform of

i. ${}^{n}C_r \; 0 \le r \le n$ ii. ${}^{n+r}C_r$ iii. $\frac{1}{(n+r)!}$ iv. $\frac{1}{(n-r)!}$

Solution: i. $Z\{{}^{n}C_r\} = \sum_{r=0}^{n} {}^{n}C_r \, z^{-r}$

$$= 1 + {}^{n}C_1 \, z^{-1} + {}^{n}C_2 \, z^{-2} + \cdots + {}^{n}C_n \, z^{-n}$$

$$= (1 + z^{-1})^n$$

ii. $Z\{{}^{n+r}C_r\} = \sum_{r=0}^{\infty} {}^{n+r}C_r \, z^{-r}$

$$= 1 + {}^{n+1}C_1 \, z^{-1} + {}^{n+2}C_2 \, z^{-2} + {}^{n+3}C_3 \, z^{-3} + \cdots$$

$$= 1 + (n+1)\, z^{-1} + \frac{(n+2)(n+1)}{2!} \, z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} \, z^{-3} + \cdots$$

$$= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!} \, (-z^{-1})^{-2}$$

$$+ \frac{(-n-1)(-n-2)(-n-3)}{3!} \, (-z^{-1})^{-3} + \cdots$$

$$= (1 - z^{-1})^{-n-1}$$

Example 7 Find the Z–transform of

i. $\frac{1}{n!}$ ii. $\frac{1}{(n+r)!}$ iii. $\frac{1}{(n-r)!}$

i. $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \cdots$

$$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots = e^{\frac{1}{z}}$$

$\Rightarrow Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

ii. $Z\left\{\frac{1}{(n+r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+r)!} z^{-n}$

Now $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from left shifting property $Z\{u_{n+k}\} = z^k \left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$

$\therefore Z\left\{\frac{1}{(n+r)!}\right\} = z^r \left[e^{\frac{1}{z}} - 1 - \frac{1}{z} - \frac{1}{2!\,z^2} - \cdots - \frac{1}{(r-1)!z^{r-1}}\right]$

In particular $\left\{\frac{1}{(n+1)!}\right\} = z^1 \left[e^{\frac{1}{z}} - 1\right]$

$\left\{\frac{1}{(n+2)!}\right\} = z^2 \left[e^{\frac{1}{z}} - 1 - \frac{1}{z}\right]$

iii. $Z\left\{\frac{1}{(n-r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n-r)!} z^{-n}$

Now $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from right shifting property, $Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$, k is positive integer

$$\therefore Z\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{\frac{1}{z}}$$

In particular $\left\{\frac{1}{(n-1)!}\right\} = z^{-1} e^{\frac{1}{z}}$

$$\left\{\frac{1}{(n+2)!}\right\} = z^{-2} e^{\frac{1}{z}}$$

$$\vdots$$

Example8 Find $Z\{u_{n+2}\}$ if $Z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$

Solution: Given $Z\{u_n\} = U(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$

From left shifting property $Z\{u_{n+2}\} = z^2\left[Z\{u_n\} - u_0 - \frac{u_1}{z}\right]$ ...①

Now from initial value theorem $u_0 = \lim_{z\to\infty} U(z)$

$$= \lim_{z\to\infty}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]$$

$$= \lim_{z\to\infty}\left[\frac{1}{1-\frac{1}{z}} + \frac{\frac{1}{z}}{1+\frac{1}{z^2}}\right] = 1 + 0$$

$$\therefore u_0 = 1 \quad \text{...②}$$

Also from initial value theorem $u_1 = \lim_{z\to\infty} z[U(z) - u_0]$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \lim_{z\to\infty} z\left[\frac{2z^2-z+1}{(z^2+1)(z-1)}\right] = 2$$

$$\therefore u_1 = 2 \quad \text{...③}$$

Using ②and ③ in ①, we get $Z\{u_{n+2}\} = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$

$$\Rightarrow Z\{u_{n+2}\} = \frac{z(z^2-z+2)}{(z-1)(z^2+1)}$$

Example9 Verify convolution theorem for $u_n = n$ and $v_n = 1$

Solution: Convolution theorem states that $Z\{u_n * v_n\} = U(z).V(z)$

We know that $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

$$= \sum_{m=0}^{n} m\,.\,1$$

$$= 0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\Rightarrow Z\{u_n * v_n\} = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} z^{-n} = \frac{1}{2}\left[\sum_{n=0}^{\infty} n^2 z^{-n} + \sum_{n=0}^{\infty} n\, z^{-n}\right]$$

$$= \frac{1}{2}\left[Z\{n^2\} + Z\{n\}\right]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2}\right] = \frac{1}{2}\left[\frac{z(z+1)+z(z-1)}{(z-1)^3}\right] = \frac{1}{2}\left[\frac{2z^2}{(z-1)^3}\right]$$

$$\therefore Z\{u_n * v_n\} = \frac{z^2}{(z-1)^3} \qquad \ldots ①$$

Also $U(z) = Z\{n\} = \frac{z}{(z-1)^2}$ and $V(z) = Z\{1\} = \frac{z}{z-1}$

$$\Rightarrow U(z).V(z) = \frac{z^2}{(z-1)^3} \qquad \ldots ②$$

From ①and ② $Z\{u_n * v_n\} = U(z).V(z)$

Example10 If $u_n = \delta(n) - \delta(n-1)$, $v_n = 2\delta(n) + \delta(n-1)$, Find the Z-transform of their convolution.

Solution: $U(z) = Z\{\delta(n) - \delta(n-1)\}, V(z) = Z\{2\delta(n) + \delta(n-1)\}$

Now $Z\{\delta(n)\} = 1$ and $Z\{\delta(n-1)\} = z^{-1}$ $\quad\quad \because Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$

$\therefore U(z) = 1 - z^{-1}$ and $V(z) = 2 + z^{-1}$

Also $Z\{u_n * v_n\} = U(z).V(z)$

$\Rightarrow Z\{u_n * v_n\} = (1 - z^{-1})(2 + z^{-1}) = 2 - z^{-1} - z^{-2}$

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Inverse Z-Transform

Given a function $U(z)$, we can find the sequence u_n by one of the following methods

- Inspection (Direct inversion)
- Direct division
- Partial fractions
- Residues (Inverse integral)
- Power-Series
- Convolution theorem