Multiple Integrals

The multiple integrals are the integrals of a function of two or more variables over a region in the plane or space.

Double Integrals :-

(1) If a function $f(x,y)$ is defined on a rectangular region R

$R: \ a \leq x \leq b \ \& \ c \leq y \leq d$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

(2) If a function $f(x,y)$ is defined on a region R

$R: \ a \leq x \leq b \ \& \ f_1(x) \leq y \leq f_2(x)$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$$

(3) If a function $f(x,y)$ is defined on a region R

$R: \ g_1(x) \leq x \leq g_2(x) \ \& \ c \leq y \leq d$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{g_1(x)}^{g_2(x)} f(x,y)\, dx \cdot dy$$

Email (hussein.kadhim@uomus.edu.iq)

Ex:– Calculate $\iint_R f(x,y)\,dA$ for

$f(x,y) = 1-6x^2y$ & $R: 0\le x\le 2, -1\le y\le 1$

Solution:–

$$\iint_R f(x,y)\,dA = \int_0^2\int_{-1}^{1}(1-6x^2y)\,dy\,dx = \int_0^2 \left[y - \frac{6x^2y^2}{2}\right]_{-1}^{1} dx$$

$$= \int_0^2 \left[\left(1-\frac{6x^2*1^2}{2}\right)-\left(-1-\frac{6x^2(-1)^2}{2}\right)\right]dx$$

$$= \int_0^2 [1-3x^2-(-1-3x^2)]\,dx = \int_0^2 2\,dx = 2x\Big]_0^2$$

$$= (2*2)-(2*0) = \boxed{4}$$

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Ex:– calculate $\int_0^1\int_0^y e^{y^2}\,dx\cdot dy$.

Solution:–

$$\int_0^1\int_0^y e^{y^2}\,dx\cdot dy = \int_0^1 e^{y^2}\cdot x\Big]_0^y dy = \int_0^1 [(e^{y^2}*y)-(e^{y^2}*0)]\cdot dy$$

$$= \int_0^1 y e^{y^2}dy * \frac{2}{2} = \frac{1}{2}e^{y^2}\Big]_0^1 = \frac{1}{2}(e^{1^2}-e^{0})$$

$$= \boxed{\frac{1}{2}(e^1-1)}$$

Ex:– Find $\int_0^2\int_{y^2}^{6-y} dx\cdot dy$

Solution:–

$$\int_0^2\int_{y^2}^{6-y} dx\cdot dy = \int_0^2 x\Big]_{y^2}^{6-y} dy = \int_0^2 (6-y-y^2)\,dy = 6y-\frac{y^2}{2}-\frac{y^3}{3}\Big]_0^2$$

$$= \left(6*2-\frac{2^2}{2}-\frac{2^3}{3}\right)-(0) = 12-2-\frac{8}{3}$$

$$= \frac{22}{3} = \boxed{7.33}$$

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Ex:- Calculate $\iint_R f(x,y)\,dA$ for Quiz_17-11-2024

$f(x,y) = 4x+2$ & $R: x^2 \le y \le 2x,\ 0 \le x \le 2$

Solution:-

$$\iint_R f(x,y)\,dA = \int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx = \int_0^2 (4xy+2y)\Big]_{y=x^2}^{y=2x} \cdot dx$$

$$= \int_0^2 [(4x*2x+2*2x) - (4x*x^2+2*x^2)]\cdot dx$$

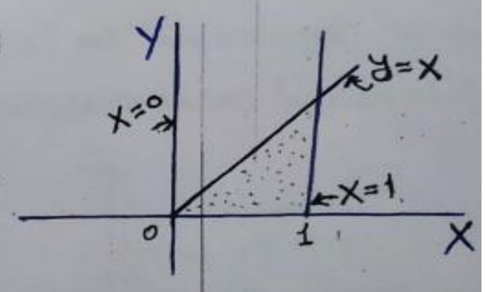
$$= \int_0^2 [8x^2+4x-(4x^3+2x^2)]dx = \int_0^2 (6x^2+4x-4x^3)\cdot dx$$

$$= \frac{6x^3}{3}+\frac{4x^2}{2}-\frac{4x^4}{4}\Big]_0^2 = 2x^3+2x^2-x^4\Big]_0^2$$

$$= 2(2)^3+2(2)^2-(2)^4 = 16+8-16 = \boxed{8}$$

Ex:- Evaluate the $\iint_R \frac{\sin x}{x}\,dA$ where R is the triangle in xy-plane bounded by the x-axis the line $x=y$ and the line $x=1$.

Solution:-



$$\int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x}y\Big]_0^{y=x}\cdot dx$$

$$\int_0^1 \frac{\sin x}{\cancel{x}}\cdot\cancel{x}\,dx = -\cos x\Big]_0^1$$

$$= -(\cos(1)-\cos(0))$$

$$= \boxed{0.459}$$

Email (hussein.kadhim@uomus.edu.iq)

Determining the Limits of Integration :-

If we want to evaluate $\iint_R f(x,y)\,dA$ over the region R shown in Fig.(1), integrating first with respect to (y) and then with respect to (x). we take the following steps:

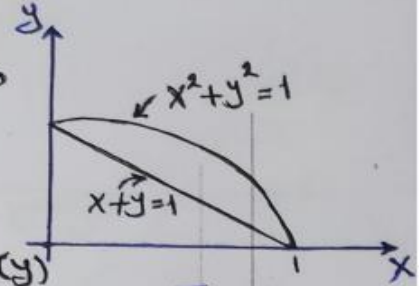


Fig.(1)

(1) we imagine a vertical line L cutting through R in the direction of increasing (y) as shown in Fig.(2).

(2) We integrate from the y-value where L enters R to the y-value where L leave R.

(3) We choose x-limits that include all the vertical lines that pass through R.

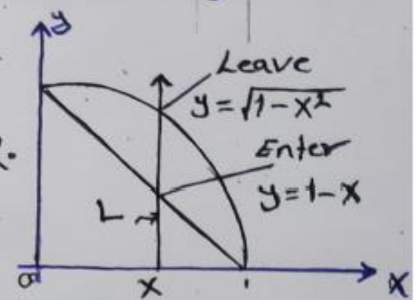


Fig.(2)

$$\therefore \iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

To calculate the same double integral as an integral with the order of integration reversed the procedure uses horizontal line as shown in Fig(3)

$$\therefore \iint_R f(x,y)\,dA = \int_{y=0}^{y=1}\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

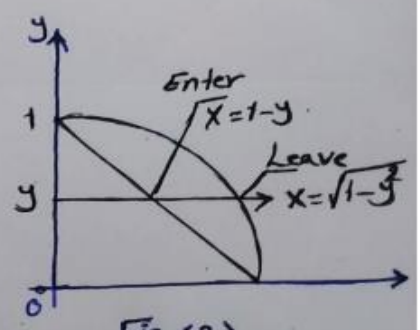


Fig (3)

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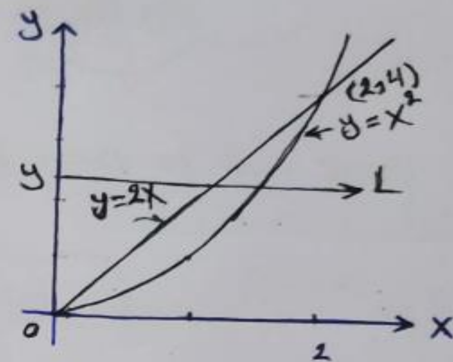
Ex:- Find an equivalent integral to the $\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx$ by the order of the integral reversed.

Solution:-

R: $x=0$ to $x=2$

$y=x^2$ to $y=2x$



$$\therefore \int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx = \int_0^4\int_{y/2}^{\sqrt{y}}(4x+2)\,dx\,dy$$

$$\int_0^4\int_{y/2}^{\sqrt{y}}(4x+2)\,dx\,dy = \int_0^4 \left(\frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}}+2x\right)\Big]_{y/2}^{\sqrt{y}}\cdot dy$$

$$= \int_0^4 \left[\left(2(\sqrt{y})^2+2\sqrt{y}\right)-\left(2\left(\frac{y}{2}\right)^2+2*\frac{y}{2}\right)\right]dy$$

$$= \int_0^4\left(2y+2\sqrt{y}-\frac{y^2}{2}-y\right)dy = \int_0^4\left(y+2\sqrt{y}-\frac{y^2}{2}\right)dy$$

$$= \left[\frac{y^2}{2}+\frac{2(y)^{3/2}}{3/2}-\frac{y^3}{2*3}\right]_0^4$$

$$= \left(\frac{4^2}{2}+\frac{4}{3}(4)^{3/2}-\frac{(4)^3}{6}\right)$$

$$= 8+\frac{4}{3}(8)-\frac{32}{3}$$

$$= \boxed{8}$$

Email (hussein.kadhim@uomus.edu.iq)

Areas, Moments and Center of Mass :-

The area of a closed bounded plane region R is:

$$A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx$$

To find the moments and center of mass of thin sheet or plate we use the formulas :-

(1) Mass :- $M = \iint \delta(x,y) \cdot dA$

where $\delta(x,y)$ is the density

(2) First moments :-

$$M_x = \iint y \cdot \delta(x,y) \cdot dA$$
$$M_y = \iint x \cdot \delta(x,y) \cdot dA$$

(3) Center of mass :-

$$\bar{x} = M_y / M$$
$$\bar{y} = M_x / M$$

(4) Moment of inertia :-

- About x-axis $I_x = \iint y^2 \cdot \delta(x,y) \cdot dA$
- About y-axis $I_y = \iint x^2 \cdot \delta(x,y) \cdot dA$
- About the origin $I_o = \iint (x^2+y^2) \cdot \delta(x,y) \cdot dA$

(5) Radii of gyration :-

- About x-axis $R_x = \sqrt{I_x / M}$
- About y-axis $R_y = \sqrt{I_y / M}$
- About the origin $R_o = \sqrt{I_o / M}$

Email (hussein.kadhim@uomus.edu.iq)

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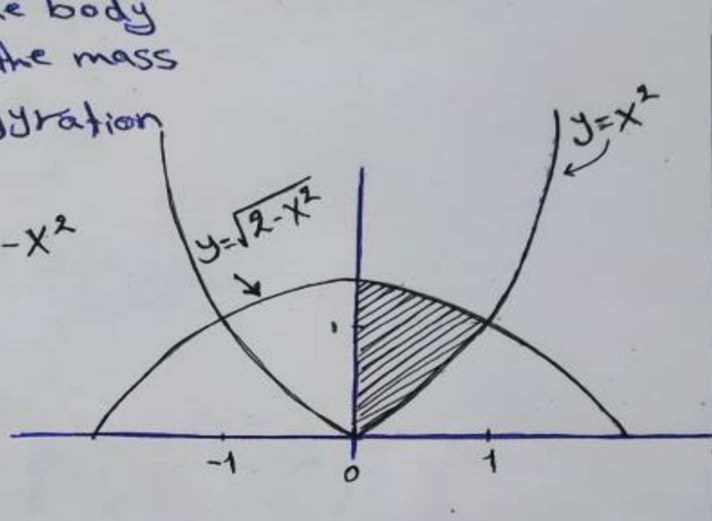
Ex: A thin plate covers the plane region R bounded by $y = x^2$, $y = \sqrt{2-x^2}$ and y-axis and with a density function $\delta = xy$

a- Find the mass of the body

b- Find the center of the mass

c- Find the radii of gyration



Solution:

$$x^2 = \sqrt{2-x^2} \Rightarrow x^4 = 2 - x^2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0$$

$$x^2 - 1 = 0 \rightarrow x^2 = 1$$

$$\therefore \boxed{x = \mp 1}$$

$$x^2 + 2 = 0 \Rightarrow x^2 = -2 \quad \text{(rejected)}$$

$$\therefore \boxed{y = 1^2 = 1}$$

a/

$$M = \iint_R \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy \cdot dy\, dx$$

$$= \int_0^1 x \frac{y^2}{2}\Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x\left[\left(\frac{(\sqrt{2-x^2})^2}{2}\right) - \frac{(x^2)^2}{2}\right] dx$$

$$= \int_0^1 x \cdot \left[\frac{2-x^2}{2} - \frac{x^4}{2}\right] dx$$

$$= \int_0^1 \left[x - \frac{x^3}{2} - \frac{x^5}{2}\right] dx = \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{12}\Big]_0^1$$

$$= \frac{1}{2} - \frac{1}{8} - \frac{1}{12} = \boxed{\frac{7}{24}}$$

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b/ $M_x = \iint_R y \cdot \delta(x,y) \cdot dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} \cdot x y^2 \cdot dy\, dx$

$= \int_0^1 x \frac{y^3}{3} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx$

$= \int_0^1 x \cdot \left[\left(\frac{(\sqrt{2-x^2})^3}{3} \right) - \frac{(x^2)^3}{3} \right] \cdot dx$

$= \int_0^1 \left[\frac{x(2-x^2)^{3/2}}{3} - \frac{x^7}{3} \right] \cdot dx$

$= \frac{-1}{2*3} \cdot \frac{(2-x^2)^{5/2}}{5/2} - \frac{x^8}{8*3} \Big]_0^1 =$

$= \left[\left(\frac{-1}{15} (2-1)^{5/2} - \frac{1^8}{24} \right) \right] - \left[\left(\frac{-1}{15} (2-0^2)^{5/2} \right) - \frac{0^8}{24} \right]$

$= \boxed{0.268}$

$\therefore \bar{y} = M_x / M = \frac{0.268}{7/24} = \boxed{0.922}$

$M_y = \iint_R x \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^2 y \cdot dy \cdot dx$

$= \int_0^1 x^2 \frac{y^2}{2} \Big]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 x^2 \left[\frac{(\sqrt{2-x^2})^2}{2} - \frac{(x^2)^2}{2} \right] \cdot dx$

$= \int_0^1 x^2 \left[\frac{2-x^2}{2} - \frac{x^4}{2} \right] \cdot dx = \int_0^1 \left(x^2 - \frac{x^4}{2} - \frac{x^6}{2} \right) \cdot dx$

$= \left(\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14} \right]_0^1 = \frac{1}{3} - \frac{1}{10} - \frac{1}{14} = \boxed{\frac{17}{105}}$

$\therefore \bar{x} = M_y / M = \frac{17/105}{7/24} = \boxed{0.56}$

$\boxed{8}$

Email (hussein.kadhim@uomus.edu.iq)

c/ $I_x = \iint_R y^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} \cdot x y^3 \cdot dy \cdot dx$

$= \int_0^1 x \left.\frac{y^4}{4}\right]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x}{4}\left[(\sqrt{2-x^2})^4 - (x^2)^4\right] \cdot dx$

$= \int_0^1 \left[\frac{x}{4}(2-x^2)^2 - \frac{x^9}{4}\right] dx = \left.\frac{-1}{4*2}\frac{(2-x^2)^3}{3} - \frac{x^{10}}{40}\right]_0^1$

$= \left[\frac{-1}{24}(2-1^2)^3 - \frac{1}{40}\right] - \left[\frac{-1}{24}(2-0)^3 - \frac{0}{40}\right]$

$= \frac{-1}{24} - \frac{1}{40} + \frac{8}{24} = \frac{4}{15} = \boxed{0.27}$

$\therefore R_x = \sqrt{I_x / M} = \sqrt{\frac{4/15}{7/24}} = \boxed{0.96}$

$I_y = \iint_R x^2 \cdot \delta(x,y)\, dA = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x^3 y \cdot dy \cdot dx$

$= \int_0^1 x^3 \left.\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} \cdot dx = \int_0^1 \frac{x^3}{2}\left[(\sqrt{2-x^2})^2 - (x^2)^2\right] \cdot dx$

$= \int_0^1 \left[x^3 - \frac{x^5}{2} - \frac{x^7}{2}\right] dx = \left.\frac{x^4}{4} - \frac{x^6}{12} - \frac{x^8}{16}\right]_0^1$

$= \frac{1}{4} - \frac{1}{12} - \frac{1}{16} = \boxed{\frac{5}{48}}$

$\therefore R_y = \sqrt{I_y / M} = \sqrt{\frac{5/48}{7/24}} = \boxed{0.597}$

$I_o = I_x + I_y = \frac{4}{15} + \frac{5}{48} = \boxed{\frac{89}{240}}$

$R_o = \sqrt{I_o / M} = \sqrt{\frac{89/240}{7/24}} = \boxed{1.13}$

Email (hussein.kadhim@uomus.edu.iq)

Triple Integrals :-

The integral $\iiint_D f(x,y,z)\,dV$ is called the volume integral and it is evaluated in the manner that

$$\iiint_D f(x,y,z)\cdot dV = \iint_R \left(\int_{z_1}^{z_2} f(x,y,z)\,dz\right)\cdot dA$$

The volume of a closed bounded region D in Space is

$$V_D = \iiint_D dV$$

Ex:- Evalute $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\cdot dx\cdot dy$

Solution:-

$$\int_0^{\sqrt{2}}\int_0^{3y} z\Big]_{x^2+3y^2}^{8-x^2-y^2}\cdot dx\cdot dy = \int_0^{\sqrt{2}}\int_0^{3y}\left[(8-x^2-y^2)-(x^2+3y^2)\right]\cdot dx\cdot dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y}(8-2x^2-4y^2)\cdot dx\cdot dy = \int_0^{\sqrt{2}} 8x - 2\frac{x^3}{3} - 4y^2x\Big]_0^{3y}\cdot dy$$

$$= \int_0^{\sqrt{2}}\left[\left(8*3y - \frac{2}{3}(3y)^3 - 4y^2*3y\right)\right]dy = \int_0^{\sqrt{2}}(24y-18y^3-12y^3).dy$$

$$= \int_0^{\sqrt{2}}(24y-30y^3).dy = \frac{24y^2}{2} - \frac{30y^4}{4}\Big]_0^{\sqrt{2}}$$

$$= 12(\sqrt{2})^2 - \frac{15}{2}(\sqrt{2})^4 = 24-30$$

$$= \boxed{-6}$$

Email (hussein.kadhim@uomus.edu.iq)

H.W

(1) Evaluate the integral:

a- $\int_0^2 \int_{-1}^1 (x-y)\,dy\,dx$

b- $\int_0^1 \int_1^2 x y e^x \cdot dy\,dx$

c- $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} \cdot dy \cdot dx$

(2) Evalute the double integral over the region R

a/ $\iint_R \frac{\sqrt{x}}{y^2}\,dA$ R: $0 \leq x \leq 4$ & $1 \leq y \leq 2$

b/ $\iint_R x y \cos y \cdot dA$ R: $-1 \leq x \leq 1$ & $0 \leq y \leq \pi$

c/ $\iint_R \frac{x y^3}{x^2+1} \cdot dA$ R: $0 \leq x < 1$ & $0 \leq y \leq 2$

(3) Sketch the region of integration and write an equivalent double integral with the order of integration reversed

a/ $\int_0^1 \int_2^{4-2x} \cdot dy \cdot dx$

b/ $\int_0^1 \int_{1-x}^{1-x^2} \cdot dy \cdot dx$

c/ $\int_0^1 \int_y^{\sqrt{y}} \cdot dx \cdot dy$

(4) Sketch the region bounded by the Parabolas $x = y^2$ and $x = 2y - y^2$. Then express the region's area as an iterated double integral and evaluate the integral.

(5) Evalute $\int_0^{\ln 2} \int_0^{e^x} \cdot dy \cdot dx$

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Email (hussein.kadhim@uomus.edu.iq)

⑥ A thin Plate cover the triangle region bounded by the X-axis and the line $X=1$ and $y=2X$ in the first quarter. The plate density is $\delta(x,y) = 6X+6y+6$. find

a/ The mass of the body.

b/ The Center of mass.

c/ The radii of gyration.

⑦ Evaluate

a/ $\int_0^1 \int_0^{3-3X} \int_0^{3-3X-y} dz \cdot dy \cdot dx$

b/ $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\, dy \cdot dx$

c/ $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \cdot dx \cdot dy \cdot dz$

Email (hussein.kadhim@uomus.edu.iq)

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Email (hussein.kadhim@uomus.edu.iq)