Discrete Mathematics

Lecture 5

Sets and set operations

By

Asst. Lect. Ali Al-Khawaja

Basic discrete structures

- **Discrete math =**
    - study of the discrete structures used to represent discrete objects

- Many discrete structures are built using **sets**
    - **Sets = collection of objects**

Examples of discrete structures built with the help of sets:
- **Combinations**
- **Relations**
- **Graphs**

Set

- **<u>Definition</u>**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set. (Cantor's naive definition)

- **Examples:**
  - **Vowels in the English alphabet**

    $V = \{ a, e, i, o, u \}$
  - **First seven prime numbers.**

    $X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

Representing sets

Representing a set by:

    1) Listing (enumerating) the members of the set.

    2) Definition by property, using the set builder notation

        $\{x|\ x$ has property P$\}$.

Example:

- Even integers between 50 and 63.

    1) E = {50, 52, 54, 56, 58, 60, 62}

    2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

      • **A= {1,2,3 …, 100}**

Important sets in discrete math

- **Natural numbers:**
  - $\mathbf{N} = \{0,1,2,3, \ldots\}$

- **Integers**
  - $\mathbf{Z} = \{\ldots, -2,-1,0,1,2, \ldots\}$

- **Positive integers**
  - $\mathbf{Z^+} = \{1,2, 3\ldots\}$

- **Rational numbers**
  - $\mathbf{Q} = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$

- **Real numbers**
  - $\mathbf{R}$

Equality

Definition: Two sets are equal if and only if they have the same elements.

Example:

- $\{1,2,3\} = \{3,1,2\} = \{1,2,1,3,2\}$

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

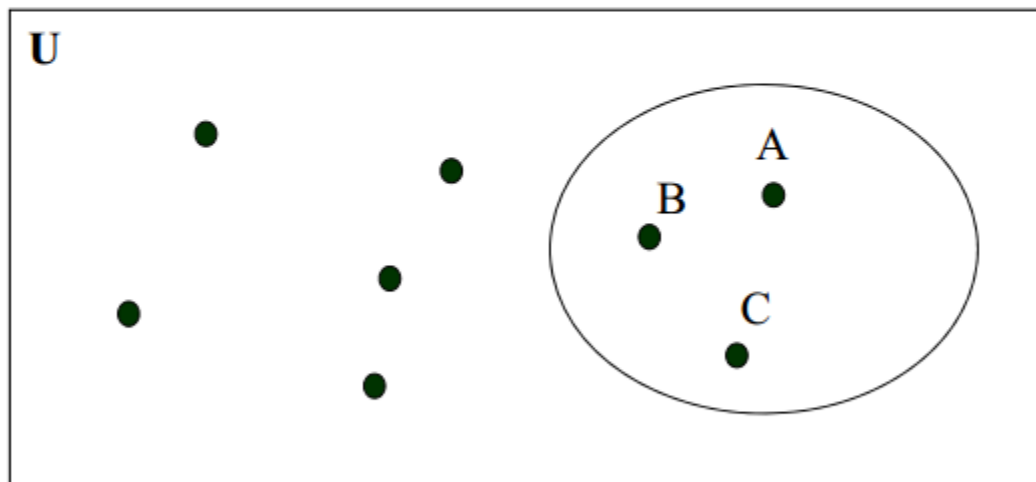**Example:** Are $\{1,2,3,4\}$ and $\{1,2,2,4\}$ equal?

No!

Special sets

- **Special sets:**

  - **The <u>universal set</u>** is denoted by **U:** the set of all objects under the consideration.

  - **The <u>empty set</u>** is denoted as ∅ **or { }.**
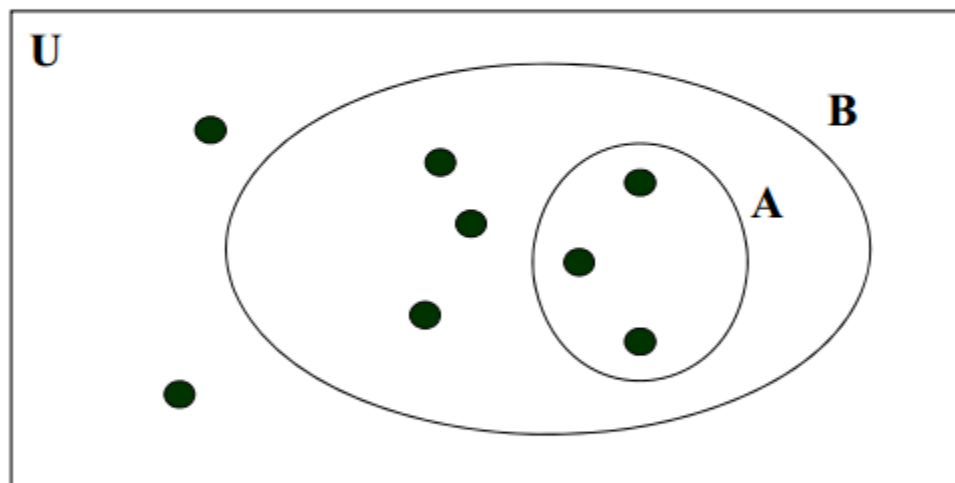
Venn diagrams

- A set can be visualized using **Venn Diagrams**:
  - V={ A, B, C }

A Subset

- **<u>Definition</u>**: A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use $A \subseteq B$ to indicate **A is a subset of B**.

- Alternate way to define A is a subset of B:

$$\forall x \, (x \in A) \rightarrow (x \in B)$$

Empty set/Subset properties

Theorem $\varnothing \subseteq S$

- **Empty set is a subset of any set.**

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x \, (x \in A \rightarrow x \in B)$.

- We must show the following implication holds for any S

   $\forall x \, (x \in \varnothing \rightarrow x \in S)$

- Since the empty set does not contain any element, $x \in \varnothing$ is **always False**

- Then the implication is **always True.**

End of proof

Subset properties

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x \, (x \in A \rightarrow x \in B)$.

- Applying this to S we get:

- $\forall x \, (x \in S \rightarrow x \in S)$ which is trivially **True**

- End of proof

Note on equivalence:
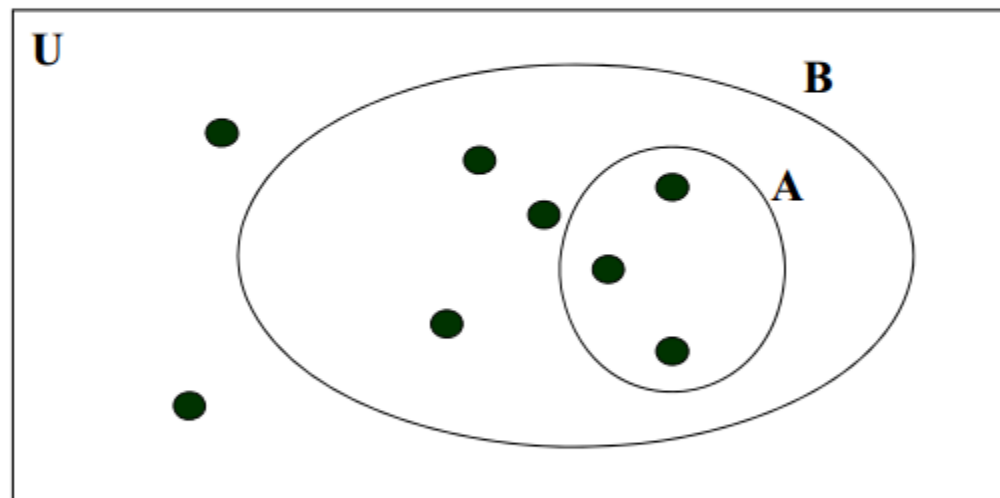
- Two sets are equal if each is a subset of the other set.

A proper subset

Definition: A **set** A is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.
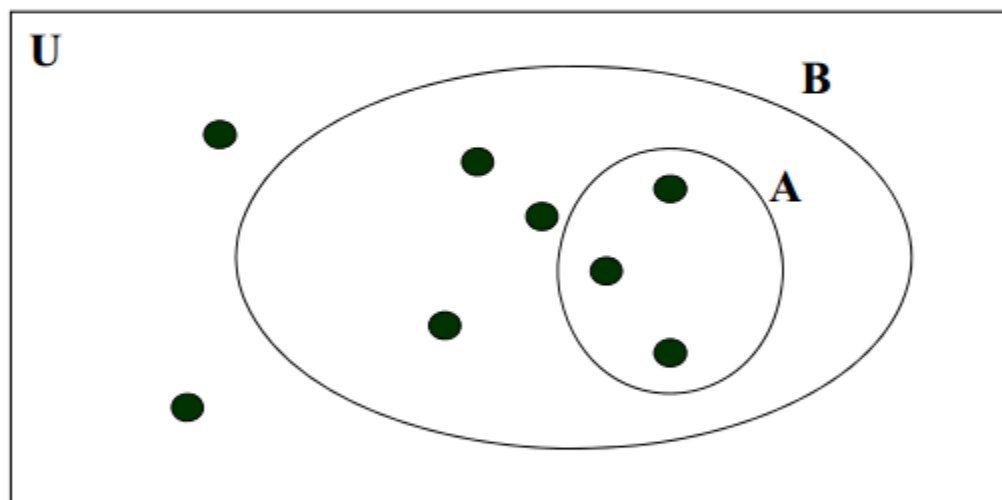
A proper subset

Definition: A **set** A is said to be a **proper subset** of B if and only if $A \subseteq B$ and $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.

Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **| S |.**

Examples:

- **V={1 2 3 4 5}**

    | V | = 5

- **A={1,2,3,4, …, 20}**

    |A| =20

- **| ∅ | = 0**

Infinite set

Definition: A set is **infinite** if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }

- The set of reals is an infinite set.

Power set

Definition: Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set $\varnothing$
- What is the power set of $\varnothing$? $P(\varnothing) = \{ \varnothing \}$
- What is the cardinality of $P(\varnothing)$? $| P(\varnothing) | = 1$.

- Assume set $\{1\}$
- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

Power set

- P({1}) = { $\emptyset$, {1} }
- |P({1})| = 2

- Assume {1,2}
- P({1,2}) = { $\emptyset$, {1}, {2}, {1,2} }
- |P({1,2})| = 4

- Assume {1,2,3}
- P({1,2,3}) = {$\emptyset$, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

- **If S is a set with |S| = n then | P(S) | = ?**

N-tuple

- Sets are used to represent unordered collections.
- **Ordered-n tuples** are used to represent an ordered collection.

Definition: An **ordered n-tuple** $(x_1, x_2, ..., x_N)$ is the ordered collection that has x_1 as its first element, x_2 as its second element, ..., and x_N as its N-th element, $N \geq 2$.

Example:

- Coordinates of a point in the 2-D plane $(12, 16)$

Cartesian product

<u>Definition</u>: Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where s $\in$ S and t $\in$ T. Hence,

- $\qquad$ S x T = { (s,t) | s $\in$ S $\wedge$ t $\in$ T}.

Examples:

- $\qquad$ S = {1,2} and T = {a,b,c}
- $\qquad$ S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- $\qquad$ T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: S x T $\neq$ T x S **!!!!**

Cardinality of the Cartesian product

- $|S \times T| = |S| * |T|$.

Example:
- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}
- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}
- |A x B| = 9
- |A|=3, |B|=3 → |A| |B|= 9

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Set operations

Definition: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

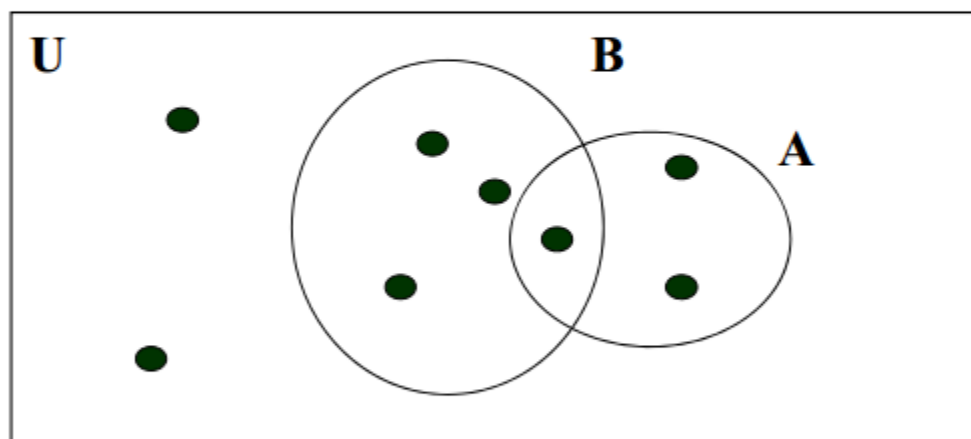- Alternate: $A \cup B = \{ x \mid x \in A \lor x \in B \}$.
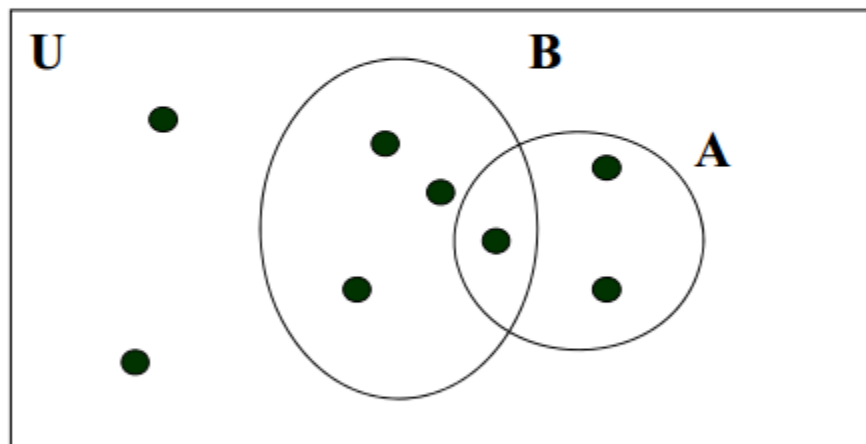
- **Example:**
- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

Set operations

Definition: Let A and B be sets. The **intersection of A and B**, denoted by A ∩ B, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \land x \in B \}$.

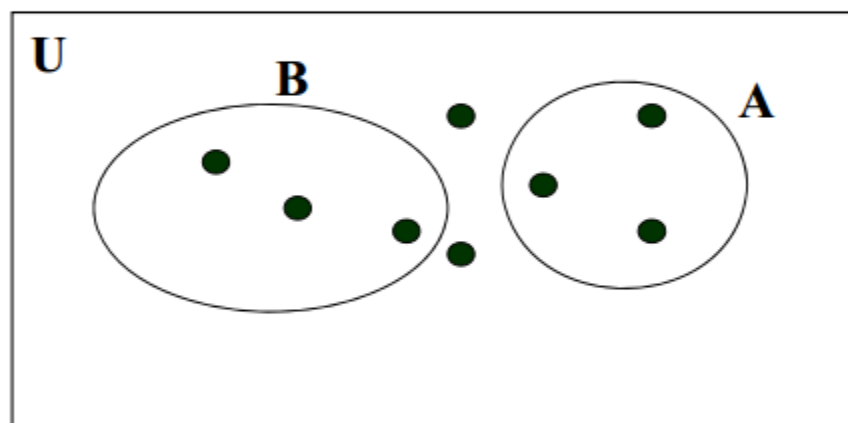Example:

- $A = \{1,2,3,6\}$ $B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$

Disjoint sets

Definition: Two sets are called **disjoint** if their intersection is empty.

- Alternate: A and B are disjoint **if and only if** $A \cap B = \varnothing$.
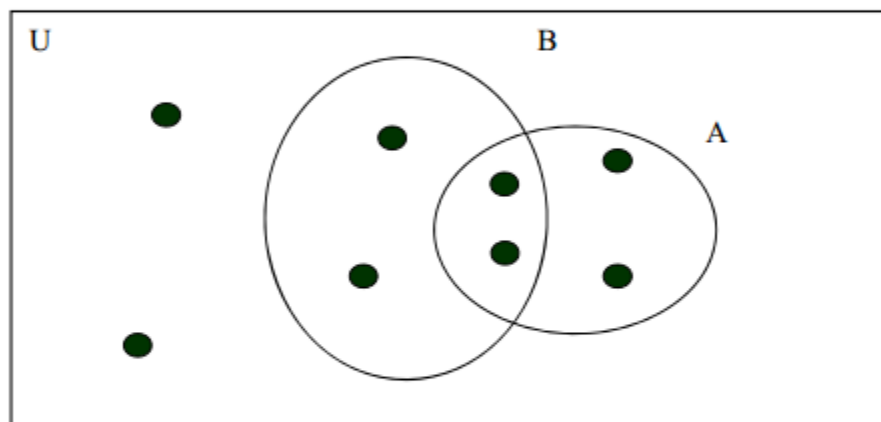

Example:

- A={1,2,3,6}   B={4,7,8}   Are these disjoint?
- Yes.
- $A \cap B = \varnothing$

Cardinality of the set union

Cardinality of the set union.

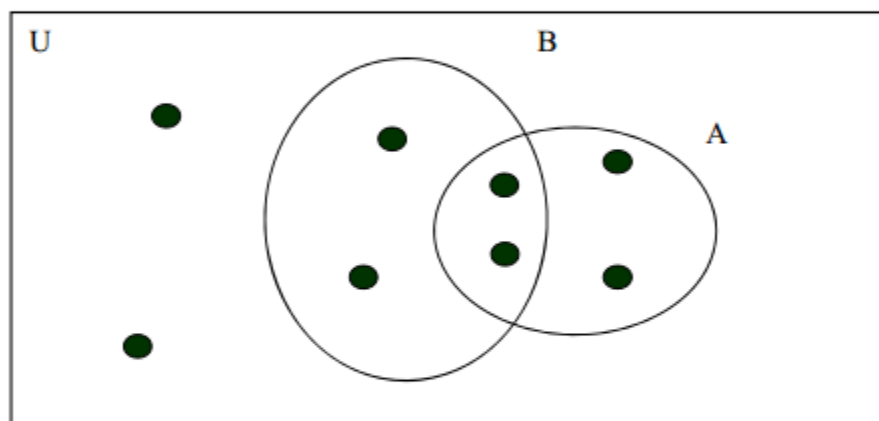- $|A \cup B| = |A| + |B| - |A \cap B|$

- Why this formula?

Cardinality of the set union

Cardinality of the set union.
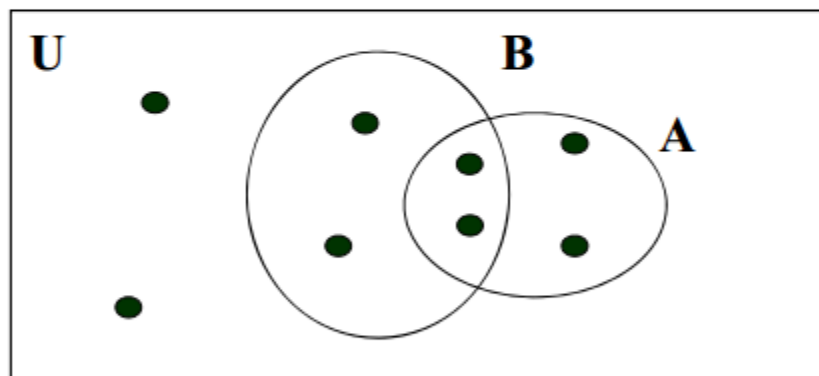
* $|A \cup B| = |A| + |B| - |A \cap B|$

* Why this formula? Correct for an over-count.

* More general rule:

    – **The principle of inclusion and exclusion.**

Set difference

Definition: Let A and B be sets. The **difference of A and B**, denoted by **A - B**, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{ x \mid x \in A \land x \notin B \}$.

Example: A= {1,2,3,5,7} B = {1,5,6,8}

- A - B ={2,3,7}

Think you

Any questions ??