2.5 Linear First Order Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$ General form of linear Equations

where:

$P(x)$: The function adjacent to (y) when the coefficient of $\frac{dy}{dx}$ is equal to 1.

$P(x)$: هو الدالة المجاورة لل y عندما معامل $\frac{dy}{dx}$ يساوي 1.

$Q(x)$: The right side function is free from (y) and its derivatives when the coefficient of $\frac{dy}{dx}$ is equal to 1.

$Q(x)$: هو الدالة في الجهة اليمنى والخالية من المتغير y ومشتقاته عندما معامل $\frac{dy}{dx}$ يساوي 1.

To Find General Solution for Linear Equation:

let (r) is an integrating factor; then multiply the general linear equation by (r):

$$(r)\frac{dy}{dx} + (r)P(x)y = (r)Q(x)$$

let:

$$(r)\frac{dy}{dx} + (r)P(x)y = (r)Q(x) = \frac{d}{dx}(ry)$$

$$\therefore (r)\frac{dy}{dx} + (r)P(x)y = \frac{d}{dx}(ry) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$and\ (r)Q(x) = \frac{d}{dx}(ry) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

from eq. (1):

$$(r)\frac{dy}{dx}+(r)P(x)y=r\frac{dy}{dx}+y\frac{dr}{dx}$$

$$(r)P(x)=\frac{dr}{dx}\qquad\rightarrow\ re-arrangement$$

$$\frac{dr}{r}=P(x)dx\qquad\rightarrow\qquad\int\frac{dr}{r}=\int P(x)dx$$

$\ln r=\int P(x)dx$ نأخذ e للطرفين

$\therefore r=\boldsymbol{e^{\int P(x)dx}}$ .... (3)   integrating factor for linear equation

from eq. (2):

$$(r)Q(x)=\frac{d}{dx}(ry)\qquad\rightarrow\ re-arrangement$$

$$d(ry)=\ (r)Q(x)\,dx\qquad\rightarrow\qquad\int d(ry)=\int(r)Q(x)\,dx\ +c$$

$$ry=\int(r)Q(x)\,dx\ +c$$

$$\therefore y=\frac{1}{r}\left(\int(r)Q(x)\,dx\ +c\right)\text{ .... }(4)$$

Sub eq. (3) and eq. (4) to get:

$$\boxed{\therefore \boldsymbol{y=e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}\cdot Q(x)\,dx\ +c\right)}}$$

general solution for linear equaion

Example (1): Solve the equation $dy-y\,dx-x\,dx=0$?

Solve:

$$\frac{dy}{dx}+P(x)y=Q(x)\qquad\text{general form of linear equaion}$$

$$dy-y\,dx-x\,dx=0\quad\}\div dx\qquad\rightarrow\qquad\frac{dy}{dx}-y-x=0$$

$$\frac{dy}{dx}-y=x$$

$\therefore P(x) = -1, \quad Q(x) = x$

$\therefore y = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} \cdot Q(x)\, dx \, + c\right)$

$y = e^{-\int -dx}\left(\int e^{\int -dx} \cdot x\, dx \, + c\right)$

$y = e^{x}\left(\int e^{-x} \cdot x\, dx \, + c\right)$

ملاحظة: ان تكامل ضرب دالتين **اما** ان يحل بطريقة $(u \cdot dv)$ **او** طريقة الجدول. فهنا سوف نجري التكامل بالطريقة $(u \cdot dv)$.

$\int e^{-x} \cdot x\, dx$ use $(u \cdot dv)$ to solve integral

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$let\ \ u = x \quad \rightarrow \quad du = dx$

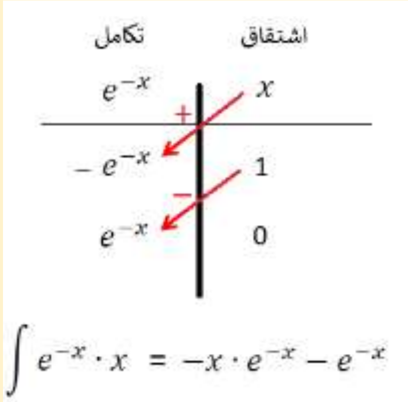
$dv = e^{-x} \cdot\ dx \quad \rightarrow \quad v = -e^{-x}$

$y = e^{x}\left(-x \cdot e^{-x} - \int -e^{-x} \cdot dx + c\right)$

$y = e^{x}(-x \cdot e^{-x} - e^{-x} + c)$

$y = -x - 1 + ce^{x}$

وبالإمكان حلها ايضا بطريقة الجدول الموضحة ادناه:



Example (2): Solve the equation $\acute{y} - 3y = 6$?

Solve:

$\frac{dy}{dx} + P(x)y = Q(x)$ general form of linear equaion

$\therefore P(x) = -3, \quad Q(x) = 6$

$\therefore y = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} \cdot Q(x)\, dx \, + c\right)$

$y = e^{-\int -3\, dx}\left(\int e^{\int -3\, dx} \cdot 6\, dx \, + c\right)$

$y = e^{3x}\left(\int e^{-3x} \cdot 6\, dx \, + c\right)$

$y = e^{3x}(-2\, e^{-3x} + c) \quad \rightarrow \quad y = -2 + c\, e^{3x}$

Problems:

H.W: Solve the equations:

1) $\acute{y} + \frac{y}{1-x} = x^2 - x$ **Answer**: $y = (1-x)\left[-\frac{x^2}{2} + c\right]$

2) $(x-y)\,dx + x\,dy = 0$ **Answer**: $y = x\,(-\ln x + c)$

3) $\acute{y} + y = e^x, \quad y(0) = 2$ **Answer**: $y = \frac{1}{2}\,(e^x + 3e^{-x})$