Ex: Find the complete solution of the differential equation.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0 \Rightarrow m_1 = m_2 = -1$ on double root

$$y_h = (c_1 x e^{-x} + c_2 e^{-x}) \Rightarrow y_h = (c_1 x + c_2)e^{-x}.$$

Since $\Rightarrow$ m as a double root $\therefore y_p = A x^2 e^{-x}$.

$y_p = A x^2 e^{-x}$

$y_p' = -A x^2 e^{-x} + 2A x e^{-x}$

$y_p'' = A x^2 e^{-x} - 2A x e^{-x} + (-2A x e^{-x} + 2A e^{-x})$

$= -4A x e^{-x} + A x^2 e^{-x} + 2A e^{-x}$.

sub into equation $\rightarrow$

$$(-4Axe^{-x} + Ax^2 e^{-x} + 2Ae^{-x}) + 2(-Ax^2e^{-x} + 2Axe^{-x}) + Ax^2e^{-x} = e^{-x}$$

$$-4Axe^{-x} + Ax^2e^{-x} + 2Ae^{-x} - 2Ax^2e^{-x} + 4Axe^{-x} + Ax^2e^{-x} = e^{-x}.$$

$\therefore 2Ae^{-x} = e^{-x} \Longrightarrow A = 1/2 .$

$$y_p = A x^2 e^{-x}$$
$$= \frac{1}{2} x^2 e^{-x}$$

$$\therefore y = y_h + y_p$$
$$= c_1 x e^{-x} + c_2 e^{-x} + \frac{1}{2} x^2 e^{-x}$$

$$\boxed{y = e^{-x}\left(c_1 x + c_2 + \frac{x^2}{2}\right)}$$

الحالة الثانية :

→ If $R(x)$ has a term that is a constant multiple of $\sin kx$, $\cos kx$

→ if k is not a root of the characteristic equation then

$$y_p = B \cos kx + C \sin kx$$

Ex: Find the complete solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 5 \sin 2x$$

$$\Longrightarrow \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow m^2 + 4m + 3 = 0 \Longrightarrow (m+1)(m+3) = 0$$

$\Longrightarrow m_1 = -1 , \ m_2 = -3$

$\therefore y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y_h = c_1 e^{-x} + c_2 e^{-3x}$

Let $y_p = A \sin 2x + B \cos 2x$

$y_p' = 2A \cos 2x - 2B \sin 2x$

$y_p'' = -4A \sin 2x - 4B \cos 2x$

$\Rightarrow (-4A \sin 2x - 4B \cos 2x) + 4(2A \cos 2x - 2B \sin 2x) + 3(A \sin 2x + B \cos 2x) = 5 \sin 2x .$

$(-A - 8B) \sin 2x + (-B + 8A) \cos 2x = 5 \sin 2x$

$$\left.\begin{array}{l} -A - 8B = 5 . \\ -B + 8A = 0 . \end{array}\right\} \Longrightarrow A = -\frac{1}{13} , \ B = -\frac{8}{13}$$

$\therefore y = y_p + y_h$

$= -\frac{1}{13} \sin 2x - \frac{8}{13} \cos 2x + c_1 e^{-x} + c_2 e^{-3x} .$

Fourier Series

الصيغه العامه لمتسلسلة فورير هي :

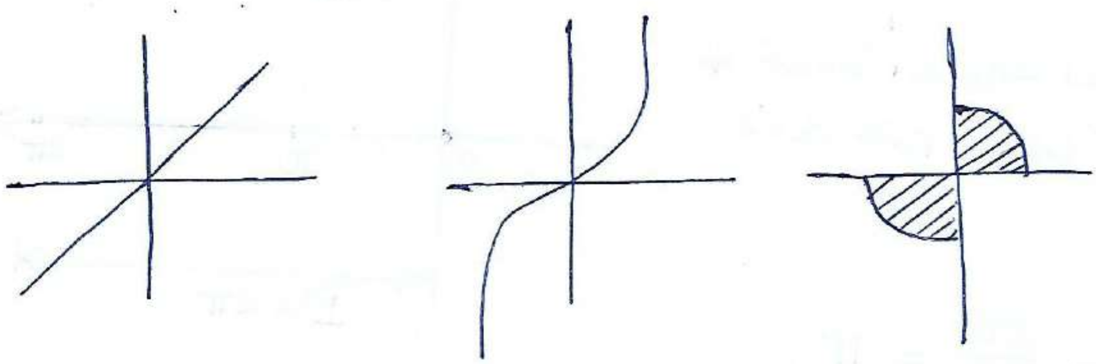
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$a_0 = \frac{1}{p} \int_{b_1}^{b_2} f(x)\, dx.$$

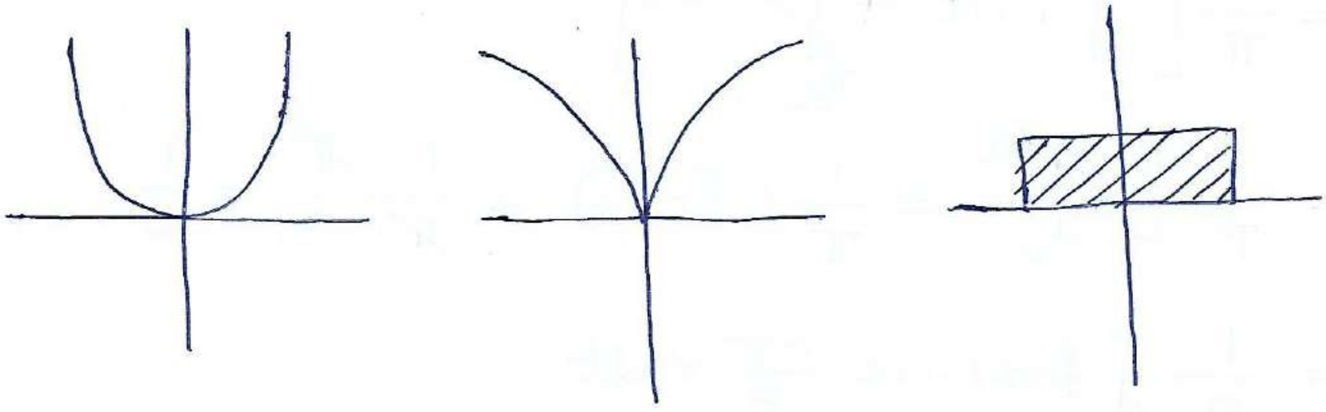
$$a_n = \frac{1}{p} \int_{b_1}^{b_2} f(x) \cos \frac{n\pi}{p} x \, dx.$$

$$b_n = \frac{1}{p} \int_{b_1}^{b_2} f(x) \sin \frac{n\pi}{p} x \, dx.$$

* الدالة الفردية : هي الدالة التي عند رسمها تكون متناظرة حول نقطه الأصل كما في الامثله الاتيه :



* الداله الزوجية : هي الداله التي عند رسمها تكون متناظرة حول محور (y) كما في الامثلة الاتيه :



ملاحظة مهمة جداً :

(1) الدالة الفردية (odd)

$$a_0 = a_n = 0$$
$$b_n = \checkmark$$

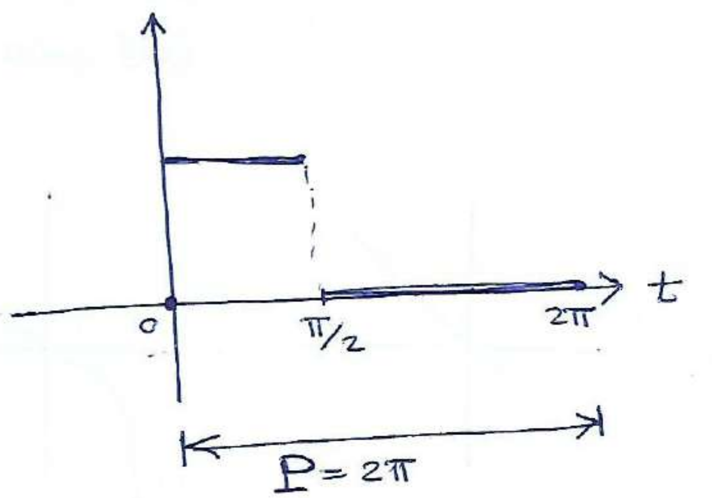
(2) الدالة الزوجية (even)

$$a_0 = a_n = \checkmark$$
$$b_n = 0$$

(٢) في حاله متسلسلة فورير تظهر تكاملات يجب حلها بطريقة التجزئة.

$$\int u\, dv = u.v - \int v . du.$$

Ex:- $f(t) = \begin{cases} 1 & 0 < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t < 2\pi \end{cases}$



* نلاحظ ان الدالة لا فردية ولا زوجية لذلك نقوم بحساب كل من a_0, a_n, b_n.

$$p = \frac{P}{2} = \frac{\cancel{2}\pi}{\cancel{2}} = \pi .$$

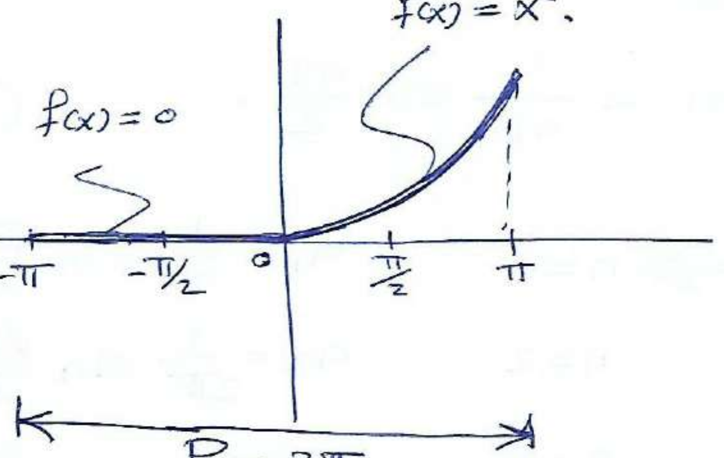
$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi/2} 1\, dt + \cancel{\int_{\pi/2}^{2\pi} 0\, dt}\right]$$

$$a_0 = \frac{1}{\pi}\left[t\right]_0^{\pi/2} = \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{\cancel{\pi}} . \frac{\cancel{\pi}}{2} = \frac{1}{2} .$$

$$a_n = \frac{1}{p}\int f(t) . \cos \frac{n\pi}{p} t\, dt$$

E(x): Find the fourier series for the functions:

$$f(x) = \begin{bmatrix} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{bmatrix}$$



$$p = \frac{P}{2} = \pi$$

$$a_0 = \frac{1}{p}\left[\int_0^\pi x^2 dx + \int_{-\pi}^0 0\, dx\right]$$

$$= \frac{1}{\pi}\int_0^\pi x^2\, dx$$

$$\therefore a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{1}{\pi}\left(\frac{\pi^3}{3} - 0\right) = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{p}\int_0^\pi x^2 \cdot \cos\frac{n\pi}{p}x\, dx = \underbrace{\frac{1}{\pi}\int_0^\pi x^2 \cos nx\, dx}$$

تكامل بواسطة $u.dv$

$$u = x^2 \qquad dv = \cos nx\, dx$$

$$du = 2x\, dx \qquad v = \frac{1}{n}\sin nx$$

$$= u.v - \int v\, du$$

$$\therefore a_n = \frac{1}{\pi}\left[x^2 \cdot \frac{1}{n}\sin nx - \frac{1}{n}\underbrace{\int \sin nx \cdot 2x\, dx}\right]_0^\pi$$

نعمل ايضا تكامل بواسطة $u.dv$

$$\therefore a_n = \frac{1}{\pi}\left[x^2 \cdot \frac{1}{n}\sin nx - \frac{2}{n}\left(x \cdot \frac{-1}{n}\cos nx\right) + \frac{1}{n}\int \cos nx\, dx\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi\right) - \left(0 + 0 - \frac{2}{n^3}\sin\ldots\right.$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} 1 \cdot \cos \frac{n\cancel{\pi}}{\cancel{\pi}} t \, dt = \frac{1}{\pi}\int_0^{\pi/2} \cos nt \, dt$$

$$= \frac{1}{n\pi}\left[\sin nt\right]_0^{\pi/2} = \frac{1}{n\pi}\left[\sin n\frac{\pi}{2} - \cancel{\sin n(0)}^{\,0}\right]$$

حل التكامل

$$a_n = \frac{1}{n\pi}\sin\frac{n\pi}{2}. \qquad (n = 1, 2, 3 \text{ ——}).$$

$$\Rightarrow n = 1 \qquad a_1 = \frac{1}{\pi}\sin\frac{\pi}{2} = \frac{1}{\pi},$$

$$n = 2 \qquad a_2 = \frac{1}{2\pi}\sin\frac{2\pi}{2} = 0,$$

$$n = 3 \qquad a_n = \frac{1}{3\pi}\sin\frac{3\pi}{2} = -\frac{1}{3\pi}$$

$$b_n = \frac{1}{P}\int f(t) \cdot \sin\frac{n\pi}{P} t \, dt$$

$$= \frac{1}{\pi}\int_0^{\pi/2} 1 \cdot \sin\frac{n\cancel{\pi}}{\cancel{\pi}} t \, dt$$

$$\therefore b_n = \frac{-1}{n\pi}\left[\cos nt\right]_0^{\pi/2} = -\frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cancel{\cos 0}^{\,1}\right]$$

$$\therefore b_n = \frac{-1}{n\pi}\left[\cos\frac{\pi n}{2} - 1\right].$$

$$\Rightarrow n = 1 \Rightarrow b_n = \frac{-1}{\pi}\left(\cos\frac{\pi}{2} - 1\right) = \frac{1}{\pi}.$$

$$n = 2 \Rightarrow b_2 = \frac{-1}{2\pi}\left(\cos\frac{2\pi}{2} - 1\right) = \frac{1}{\pi}.$$

$$n = 3 \Rightarrow b_3 = \frac{-1}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right) = \frac{1}{3\pi}.$$

$$\therefore f(t) = \frac{1}{4} + \frac{1}{\pi}\cos t - \frac{1}{3\pi}\cos 3t + \text{——} + \frac{1}{\pi}\sin t + \frac{1}{\pi}\sin t$$

$$a_n = \frac{1}{\cancel{\pi}} \cdot \frac{2\cancel{\pi}}{n^2} \cos n\pi$$

$$\therefore a_n = \frac{2}{n^2} \cos n\pi$$

$\Longrightarrow$ $n=1$ $\quad a_1 = 2\cos\pi = 2.$

$n=2$ $\quad a_2 = \frac{2}{4}\cos 2\pi = \frac{2}{4} = \frac{1}{2}.$

$n=3$ $\quad a_3 = \frac{2}{9}\cos 3\pi = \frac{-2}{9}.$

$n=4$ $\quad a_4 = \frac{2}{16}\cos 4\pi = \frac{2}{16} = \frac{1}{8}.$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 . \sin nx \, dx.$$

$$= \frac{1}{\pi}\left[x^2 . \frac{-1}{n}\cos nx + \frac{1}{n}\int 2x \cos nx \, dx \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{n}\cos nx + \frac{2}{n}\left(x . \frac{1}{n}\sin nx - \frac{1}{n}\int \sin nx \, dx \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos n\pi \right]_0^{\pi}$$

$$\therefore b_n = \frac{1}{\pi}\left[\left(\frac{-\pi^2}{n}\cos n\pi + \cancel{\frac{2\pi}{n^2}\sin n\pi}^{\,0} + \frac{2}{n^3}\cos n\pi \right) - \left(0 + 0 + \frac{2}{n^3}\cancel{\cos 0}^{\,1} \right) \right]$$

$$= \frac{-\pi}{n}\cos n\pi + \frac{2}{n^3\pi}\cos n\pi - \frac{2}{n^3\pi}$$

$n=1 \Rightarrow \quad b_1 = -\pi\cos\pi + \frac{2}{\pi}\cos\pi - \frac{2}{\pi} = \pi - \frac{2}{\pi} - \frac{2}{\pi}$

$$= \pi - \frac{4}{\pi}$$

$n=2 \Rightarrow b_2 = -\frac{\pi}{2}\cos 2\pi + \frac{2}{8\pi}\cos 2\pi - \frac{2}{8\pi}$

$$= -\frac{\pi}{2} + \frac{2}{8\pi} - \frac{2}{8\pi} = -\frac{\pi}{2}.$$

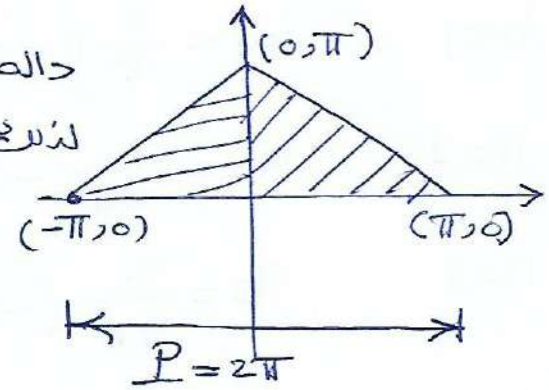
$$\therefore f(x) = \frac{\pi^2}{6} + \left(-2\cos x + \frac{1}{2}\cos 2x + \text{———}\right)$$

$$+\left(\left(\pi - \frac{4}{\pi}\right)\sin x + \left(-\frac{\pi}{2}\right)\sin 2x + \text{———}\right)$$

Ex:- Find Fourier expansion for the periodic function which definition in one period as:-

$$f(x) = \begin{cases} (\pi + x) & -\pi < x < 0 \\ (\pi - x) & 0 < x < \pi \end{cases}$$

داله متناظرة حول محور y يعني هي داله زوجية لذلك فان $b_n = 0$. هنا تكون الداله في الطرفين نفس الشكل لذلك ناخذ جزء واحد ونضربه في 2 .



$$P = \frac{P}{2} = \frac{2\pi}{2} = \pi .$$

$b_n = 0$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x)\, dx$$

$$= \frac{2}{\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} \Rightarrow a_0 = \frac{2}{\pi} \left[\left(\pi^2 - \frac{\pi^2}{2} \right) - (0 - 0) \right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \pi \cos nx \, dx - \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$a_n = \frac{2\pi}{\pi} \left[\frac{1}{n} \sin nx \right]_0^{\pi} - \frac{2}{\pi} \left[x \cdot \frac{1}{n} \sin nx - \frac{1}{n} \int \sin nx \, dx \right]_0^{\pi}$$

$$\therefore a_n = 2 \left(\frac{1}{n} \sin nx \right]_0^{\pi} - \frac{2}{\pi} \left[\frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx \right]_0^{\pi}$$

$$a_n = 2\left(\frac{1}{n}\sin n\pi - \frac{1}{n}\sin 0\right) - \frac{2}{\pi}\left[\left(\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi\right) - \cancel{\left(0+\frac{1}{n^2}\cos 0\right)}^{0}\right]$$

$$a_n = -\frac{2}{n^2\pi}\cos n\pi + \frac{2}{n^2\pi} \Rightarrow a_n = \frac{2}{n^2\pi}(-\cos n\pi + 1)$$

$n=1 \qquad a_1 = \frac{2}{\pi}(-\cos\pi + 1) = \frac{4}{\pi}.$

$n=2 \qquad a_2 = \frac{2}{4\pi}(-\cos 2\pi + 1) = 0.$

$n=3 \qquad a_3 = \frac{2}{9\pi}(-\cos 3\pi + 1) = \frac{4}{9\pi}.$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{\pi}x$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \left(\frac{4}{2}\cos nx + 0 + \frac{4}{9\pi}\cos 5x + \text{——— etc}\right.$$

Ex 8- Find the fourier expansion of the function

$$f(x) = \begin{cases} -\pi/4 & -\pi < x < 0 \\ +\pi/4 & 0 < x < \pi \end{cases}$$

$f(x) = -\frac{\pi}{4} \qquad -\pi < x < 0$

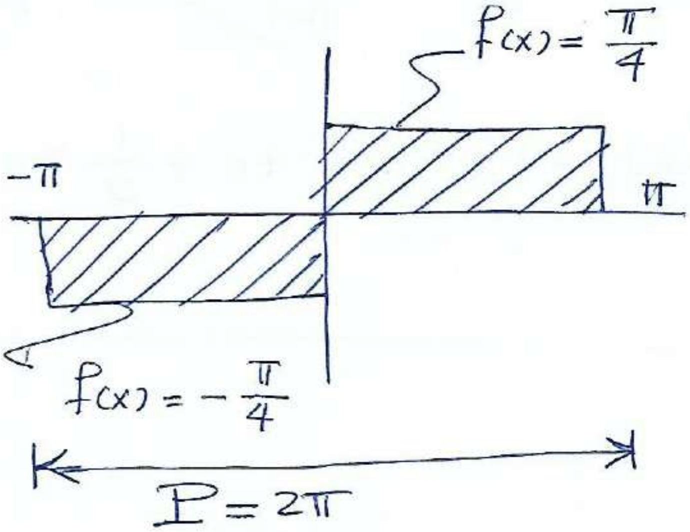
X	y	
0	$-\pi/4$	$\Rightarrow (0, -\pi/4)$
$-\pi$	$-\pi/4$	$\Rightarrow (-\pi, -\pi/4)$

$$f(x) = \frac{\pi}{4} \qquad 0 < x < \pi$$

$$x = 0 \qquad y = \frac{\pi}{4} \Rightarrow \left(0, \frac{\pi}{4}\right)$$

$$x = \pi \qquad y = \frac{\pi}{4} \Rightarrow \left(\pi, \frac{\pi}{4}\right)$$

نلاحظ ان الدالة فردية (متناظرة حول نقطة الأصل) لذلك $a_0 = a_n = 0$ فنحسب فقط b_n.



$$P = \frac{P}{2} = \frac{\cancel{2}\pi}{\cancel{2}} = \pi.$$

$$b_n = \frac{1}{P} \int f(x) \sin\left(\frac{n\pi}{P} x\right) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\frac{\pi}{4} \sin \frac{n\pi}{\pi} x \, dx + \frac{1}{\pi} \int_{0}^{\pi} \frac{\pi}{4} \sin \frac{n\pi}{\pi} x \, dx$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} \frac{\pi}{4} \sin \frac{n\pi}{\pi} x \, dx$$

$$b_n = \frac{1}{2} \int_{0}^{\pi} \sin nx \, dx$$

$$= \frac{1}{2} \left[\frac{-1}{n} \cos nx \right]_{0}^{\pi} = \frac{1}{2} \left[\left(\frac{-1}{n} \cos n\pi \right) - \left(\frac{-1}{n} \overset{1}{\cancel{\cos 0}} \right) \right]$$

$$\therefore b_n = \frac{1}{2n} \left[1 - \cos n\pi \right]$$

$$n = 1 \Rightarrow b_1 = \frac{1}{2 * 1} (1 - \cos 1 * \pi) = 1$$

$$n = 2 \Rightarrow b_2 = \frac{1}{2 * 2} (1 - \cos 2\pi) = 0$$

$$n=3 \Rightarrow b_3 = \frac{1}{3*2}(1-\cos 3\pi) = \frac{1}{3}.$$

$$n=4 \Rightarrow b_4 = \frac{1}{2*4}(1-\cos 4\pi) = 0$$

$$f(x) = \frac{a_0}{2} + \cancelto{0}{\sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$\therefore f(x) = 1 * \sin x + 0 + \frac{1}{3}\sin 3x + 0 + \text{———}$$