**دالة بيتا :** تعرف دالة بيتا كالآتي :

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; \; n > 0 \,, \quad m > 0$$

فمثلاً لإيجاد $B(2,3)$

$$B(2,3) = \int_0^1 x \;\; (1-x)^2\,dx$$

$$= \int_0^1 x(1-2x+x^2)dx$$

$$= \int_0^1 (x-2x^2+x^3)dx = \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

**ملاحظة:** $B(m,n) = B(n,m)$

**علاقة دالة بيتا بدالة كاما**

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

فمثلاً لإيجاد $B(2,3)$

$$B(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{1!\,2!}{4!} = \frac{1\times 2\times 1}{4\times 3\times 2\times 1} = \frac{1}{12}$$

**تكاملات كثيرة يمكن ايجادها باستعمال الدالتين بيتا و كاما منها :**

1. $$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n)$$

2. $$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad ; \quad 0 < p < 1$$

$\boxed{*}$ $$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$ الشكل الأول

ملاحظة: سوف نجد أن المسائل تحتوي على ثابت
زائد دالة بصورة $(a-f(x))^{??}$ مرفوعة لأي رقم.
عند الحل نأخذ الدالة $f(x)$ ونفرضها $(au)$ وتساوي
$f(x)=au$ و عند استخراج الثابت عامل مشترك
نحصل على دالة $(1-f(x))^{??}$

Ex) ①

$$\int_0^1 t^4(1-t)^3\,dt=\beta(5,4)$$

$$=\frac{\Gamma(5)\,\Gamma(4)}{\Gamma(9)}$$

$$=\frac{(4!)(3!)}{(8!)}$$

$$=\frac{6}{1680}$$

Ex (2)

$$\int_0^1 x^2\sqrt{1-x^3}\,dx = \int_0^1 (x^2)(1-x^3)^{\frac{1}{2}}\,dx$$

let $x^3 = u \Rightarrow x = u^{1/3} \Rightarrow dx = \frac{1}{3}u^{-2/3}du$

لا تنسى حدود التكامل

$(0,1) \rightarrow (0,1)$

$$\therefore \int_0^1 (u^{1/3})^2(1-u)^{\frac{1}{2}}\frac{1}{3}u^{-2/3}du$$

$$= \frac{1}{3}\int_0^1 u^{2/3}(1-u)^{\frac{1}{2}}u^{-2/3}du$$

$$= \frac{1}{3}\int_0^1 u^0(1-u)^{\frac{1}{2}}du = \frac{1}{3}\beta(1,\tfrac{3}{2})$$

$$= \frac{1}{3}\,\frac{\Gamma(1)\,\Gamma\frac{3}{2}}{\Gamma(\frac{5}{2})} = \frac{1}{3}\cdot\frac{\Gamma(\frac{3}{2})}{\frac{3}{2}\Gamma(\frac{3}{2})}$$

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

Ex) (3)

$$\int_0^2 \frac{t^2}{\sqrt{2-t}}\,dt = \int_0^2 \frac{t^2}{(2-t)^{1/2}}\,dt$$

$$= \int_0^2 t^2 (2-t)^{-1/2}\,dt$$

let $t = 2u \Rightarrow dt = 2\,du$

$(0,2) \rightarrow (0,1)$

$$\therefore = \int_0^1 (2u)^2 (2-2u)^{-1/2}\, 2\,du$$

$$= 2^2 \cdot 2^{-1/2} \cdot 2 \int_0^1 (u)^2 (1-u)^{-1/2}\,du$$

$$= 4\sqrt{2}\ \beta(3,\tfrac{1}{2}) = 4\sqrt{2}\ \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})}$$

$$= 4\sqrt{2}\ \frac{2\,\Gamma(\frac{1}{2})}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(\frac{1}{2})} = 4\sqrt{2}\cdot\frac{16}{15}$$

$$= \frac{64\sqrt{2}}{15}$$

Ex) (4)

$$\int_0^a y^4 \sqrt{a^2 - y^2}\, dy$$

$$= \int_0^a y^4 (a^2 - y^2)^{1/2} dy$$

$y^2 = a^2 u$

$y = a u^{1/2}$

$dy = a \cdot \frac{1}{2} u^{-1/2} du$

$(0, a) \rightarrow (0, 1)$

$$= \int_0^1 a^4 \cdot u^2 (a^2 - a^2 u)^{1/2} \cdot \frac{1}{2} a u^{-1/2} du$$

$$= a^4 \cdot a \cdot \frac{1}{2} a \int_0^1 u^{3/2} (1-u)^{1/2} du$$

$$= \frac{1}{2} a^6 \cdot \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2} a^6 \cdot \frac{\Gamma(\frac{5}{2})\, \Gamma(\frac{3}{2})}{\Gamma(4)} = \frac{1}{2} a^6 \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2}) \cdot \frac{1}{2} \Gamma\frac{1}{2}}{3!}$$

$$= \frac{1}{2} a^6 \frac{\frac{3}{4}\sqrt{\pi} \cdot \frac{1}{2}\sqrt{\pi}}{3 \cdot 2 \cdot 1}$$

$$= \frac{1}{2} a^6 \frac{\frac{3}{8}\pi}{6}$$

$$= \frac{18}{16} a^6 \pi$$